T.C.
İSTANBUL UNIVERSITY
INSTITUTE OF GRADUATE STUDIES IN
SCIENCE AND ENGINEERING

Ph.D. THESIS

FRACTIONAL SPACES GENERATED by POSITIVE DIFFERENTIAL OPERATORS with THE NONLOCAL CONDITION

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June 2016

FRACTIONAL SPACES GENERATED BY POSITIVE OPERATORS WITH NONLOCAL BOUNDARY CONDITIONS

Neşe NALBANT

Ph.D. Thesis – Mathematics
June 2016

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ABSTRACT

In the present work, the positivity of the differential operator with the nonlocal condition in Banach spaces is established. The structure of fractional spaces generated by this differential operator is investigated. In applications, theorems on well-posedness of local and nonlocal boundary value problems for parabolic and elliptic equations are established.

Keywords: Fractional Spaces, Positive Operators, Differential Operators, Banach Spaces,Green's Function.

YEREL OLMAYAN ŞARTLA DİFERENSİYEL POZİTİF OPERATÖRLER TARAFINDAN ÜRETİLEN KESİRLİ UZAYLAR

Neşe NALBANT

Doktora Tezi – Matematik
Haziran 2016

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ÖZ

Bu çalışmada, Banach uzayında yerel olmayan koşullu diferensiyel operatörün pozitifliği elde edildi. Bu diferensiyel operatör tarafından üretilen kesirli uzayların yapısı araştırıldı. Uygulamalarda, parabolik denklemler için yerel ve yerel olmayan sınır değer problemlerinin iyi konumlanmışlığına ait teoremler elde edildi.

Anahtar Kelimeler: Kesirli Uzaylar, Pozitif Operatörler, Diferensiyel Operatörler, Banach Uzayları, Green's Fonksiyonu.

To my family

ACKNOWLEDGEMENT

I wish to express my deepest and sincere gratitude to my thesis advisor Prof. Allaberen ASHYRALYEV for his excellent guidance, caring, patience, motivation, and providing me with an excellent atmosphere for doing research.

Thanks go to the other faculty members Prof. Feyzi BAŞAR for their valuable suggestions and comments.

I express my special thanks and appreciation to my family for their understanding, motivation and patience. Lastly, I am thankful to all colleagues and friends who made my stay at the university a memorable and valuable experience.

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CHAPTER 1

INTRODUCTION

Several problems for partial differential equations can be considered as abstract boundary value problems for first and second order ordinary differential equations in a Banach space with a densely defined unbounded space operator. The positivity of differential and difference operators in Banach spaces is important in the study of various properties of boundary value problems for elliptic, parabolic and hyperbolic partial differential equations, of stability of difference schemes for partial differential equations and summation of Fourier series converging in C-norm is well-known (see, for example, (Krein, 1968), (Fattorini, 1985), (Ashyralyev and Tetikoglu, 2015a), (Sobolevskii, 2005), (Ashyralyev and Sobolevskii, 1994), (Ashyralyev and Sobolevskii, 2004)).

The positivity of a wider class of differential and difference operators in Banach spaces has been studied by many researchers (see, for example, (Alibekov and Sobolevskii, 1977), (Ashyralyev, 1991), (Ashyralyev et al., 2014b), (Ashyralyev and Sobolevskii, 1984), (Agmon, 1996), (Solomyak, 1960), (Danelich, 1989b), (Ashyralyev, 2006), (Simirnitskii and Sobolevskii, 1981b), (Simirnitskii and Sobolevskii, 1982b), (Agmon and Nirenberg, 1963), (Agmon et al., 1959), (Agmon et al., 1964), (Alibekov, 1978), (Alibekov and Sobolevskii, 1979), (Alibekov and Sobolevskii, 1980), (Danelich, 1987a), (Danelich, 1987b), (Danelich, 1989a), (Stewart, 1980), (Solomyak, 1959), (Simirnitskiih, 1983), (Sobolevskii, 1977), (Sobolevskii, 1971), (Simirnitskii and Sobolevskii, 1982a), (Sobolevskii, 1988), (Simirnitskii and Sobolevskii, 1964), (Sobolevskii, 1997), (Sobolevskii, 1975), (Ashyraliyev, 2012), (Neginskii and Sobolevskii, 1970), (Simirnitskii and Sobolevskii, 1981a), (Ashyralyev and Agirseven, 2014a), (Ashyralyev, 2003)).

Definition 1. An operator A densely defined in a Banach space E with domain

$D(A)$ is called positive in E, if its spectrum σ_A lies in the interior of the sector of angle φ, $0 < \varphi < \pi$, symmetric with respect to the real axis, and moreover on the edges of this sector $S_1(\varphi) = \{\rho e^{i\varphi} : 0 \leq \rho \leq \infty\}$ and $S_2(\varphi) = \{\rho e^{-i\varphi} : 0 \leq \rho \leq \infty\}$, and outside of the sector the resolvent $(\lambda - A)^{-1}$ is subject to the bound (see, (Ashyralyev and Sobolevskii, 1994))

$$\left\|(A-\lambda)^{-1}\right\|_{E\to E} \leq \frac{M}{1+|\lambda|}.$$

The infimum of all such angles φ is called the spectral angle of the positive operator A and is denoted by $\varphi(A) = \varphi(A, E)$. The operator A is said to be strongly positive in a Banach space E if $\varphi(A, E) < \frac{\pi}{2}$.

Throughout the present thesis, M will indicate with positive constants which can be different from time to time and we are not interested in precise. We will write $M(\alpha, \beta, \cdots)$ to stress the fact that the constant depends only on $\alpha, \beta, \cdots$.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_{\alpha,p} = E_{\alpha,p}(E, A), (1 \leq p < \infty), E_\alpha = E_{\alpha,\infty}(E, A), 0 \leq \alpha \leq 1$ consisting of those $v \in E$ for which norms

$$\|v\|_{E_{\alpha,p}} = \left(\int\limits_0^\infty \left(\lambda^\alpha \left\|A(\lambda+A)^{-1}v\right\|_E\right)^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}},$$

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \left\|A(\lambda+A)^{-1}v\right\|_E$$

are finite, respectively.

The structure of fractional spaces generated by positive differential and difference operators and its applications to partial differential equations has been investigated by many researchers (see, for example, (Ashyralyev and Yaz, 2006), (Ashyralyev and Tetikoglu, 2014), (Tetikoğlu, 2012), (Ashyralyev and Yenial-Altay, 2005), (Ashyralyev and Agirseven, 2014b), (Ashyralyev et al., 2014a), (Ashyralyev and Kendirli, 2000), (Alibekov and Sobolevskii, 1977), (Ashyralyev and Akturk, 2015), (Triebel, 1978), (Semenova, 2012), (Ashyralyev and Yakubov, 1998), (Bazarov, 1989), (Nalbant, 2011), (Ashyralyev and Kendirli, 2001), (Ashyralyev et al., 2014c), (Ashyralyev and Karakaya, 1995), (Ashyralyev and Taskin, 2011), (Ashyralyev and Prenov, 2012), (Ashyralyev and Prenov, 2014), (Ashyralyev and Sobolevskii, 1988), (Ashyralyev and Tetikoglu, 2015b), (Tetikoğlu, 2015), (Ashyralyev and Nalbant, 2016)).

Important progress has been made in the study of positive operators from the viewpoint of the stability analysis of high order accuracy difference schemes for partial differential equations. It is well-known that the most useful methods for stability analysis of difference schemes are difference analogue of maximum principle and energy method. The application of theory of positive difference operators allows us to investigate the stability and coercive stability properties of difference schemes in various norms for partial differential equations especially when one can not use a maximum principle and energy method. However, the positivity of differential and difference operators is not well-investigated in general. Therefore, the investigation of positivity of differential and difference operators in Banach spaces and its applications to partial differential equations is an important subject. Finally, we should mention that the positivity of difference operators with nonlocal conditions is investigated only in one-dimensional case. In (Ashyralyev and Karakaya, 1995), A. Ashyralyev, I. Karakaya considered the differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{d^2u}{dx^2} + \delta u \tag{1.1}$$

with domain $D(A^x) = \left\{u \in C^{(2)}\left[0,l\right] : u\left(0\right) = u\left(l\right), u'\left(0\right) = u'\left(l\right)\right\}$. Let $a(x)$ be the smooth function defined on the segment $[0,l]$ and $a(x) \geq a > 0$. It was proved that A^x was the strongly positive operator in $C\left[0,l\right]$. For $\alpha \in (0, \frac{1}{2})$, the norms of the space $E_\alpha(C\left[0,l\right], A^x)$ and the Hölder space $C^{2\alpha}\left[0,l\right]$ were equivalent. It follows that A^x was the strongly positive operator in $C^{2\alpha}\left[0,l\right]$.

In (Ashyralyev and Kendirli, 2000)- (Ashyralyev and Kendirli, 2001), A. Ashyralyev and B. Kendirli considered the difference operator A_h^x defined by formula

$$A_h^x u^h = \left\{-a(x_k)\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k\right\}_1^{M-1}, \ u^h = \{u_k\}_0^M, Mh = l \tag{1.2}$$

with $u_0 = u_M$, $u_1 - u_0 = u_M - u_{M-1}$. This operator was a first order of approximation of the differential operator A^x defined by formula (1.1). They proved that A_h^x was the strongly positive operator in C_h. For $\alpha \in (0, \frac{1}{2})$, the norms of the space $E_\alpha(C_h, A_h^x)$ and the Hölder space $C_h^{2\alpha}$ were equivalent. It follows that A_h^x was the strongly positive operator in $C_h^{2\alpha}$.

A. Ashyralyev and N. Yenial-Altay considered in (Ashyralyev and Yenial-Altay, 2005) the difference operator defined by formula

$$A_h^x u^h = \left\{ -a(x_k) \frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k \right\}_1^{M-1}, \ u^h = \{u_k\}_0^M, Mh = l \qquad (1.3)$$

with $u_0 = u_M$, $-u_2 + 4u_1 - 3u_0 = u_{M-2} - 4u_{M-1} + 3u_M$. This operator was a second order of approximation of the differential operator A^x defined by formula (1.1). They proved that A_h^x was the strongly positive operator in C_h. For $\alpha \in (0, \frac{1}{2})$, the norms of the space $E_\alpha(C_h, A_h^x)$ and the Hölder space $C_h^{2\alpha}$ were equivalent. It follows that A_h^x was the strongly positive operator in $C_h^{2\alpha}$.

A. Ashyralyev considered in (Ashyralyev, 2006) the differential operator defined by (1.1) and difference operator A_h^x which was a second order approximation of A^x and defined by formula (1.3). He proved that A^x was the strongly positive operator in the space $L_p[0,l]$, $1 \le p < \infty$ of the all integrable functions $\varphi(x)$ defined on $[0,l]$ with the norm

$$\|\varphi\|_{L_p[0,l]} = \left(\int_0^l |\varphi(x)|^p \, dx \right)^{\frac{1}{p}}.$$

$E_{\alpha,p}(L_p[0,l], A^x) = W_p^{2\alpha}[0,l]$ for all $0 < 2\alpha < 1, 1 \le p < \infty$. Here, $W_p^\mu[0,l]$ $(0 < \mu < 1)$ was the Banach space of all integrable functions $\varphi(x)$ defined on $[0,l]$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{W_p^\mu[0,l]} = \left[\int_0^l \int_0^l \frac{|\varphi(x+y) - \varphi(x)|^p}{|y|^{1+\mu p}} dy dx + \|\varphi\|_{L_p[0,l]}^p \right]^{\frac{1}{p}}, 1 \le p < \infty.$$

This fact follows from the equality $D(A^x) = W_p^2[0,l]$ for a second order differential operator A^x in $L_p[0,l]$, $1 < p < \infty$, via the real interpolation method. The alternative method of investigation adopted in (Ashyralyev and Sobolevskii, 1994), (Ashyralyev and Sobolevskii, 2004), based on estimates of fundamental solution of the resolvent equation for the operator A^x, allows us to consider also the cases $p = 1$ and $p = \infty$. It follows that A^x was the strongly positive operator in the space $W_p^{2\alpha}[0,l]$ for all $0 < 2\alpha < 1, 1 \le p < \infty$. A_h^x was the strongly positive operator in the space $L_p = L_{p,h}$, $1 \le p < \infty$ of mesh functions $\varphi^h(x)$ defined on $[0,l]_h$ with the norm

$$\|\varphi^h\|_{L_{p,h}} = \left(\sum_{x \in [0,l]_h} |\varphi^h(x)|^p h \right)^{\frac{1}{p}}.$$

$E_{\alpha,p}(L_{p,h}, A_h^x) = W_{p,h}^{2\alpha}$ for all $0 < 2\alpha < 1, 1 \leq p < \infty$. Here, $W_{p,h}^{\mu} = W_p^{\mu}\left[0,l\right]_h$ $(0 < \mu < 1)$ was the Banach space of all mesh functions $\varphi^h(x)$ defined on $\left[0,l\right]_h$ with the norm:

$$\left\|\varphi^h\right\|_{W_{p,h}^{\mu}} = \left[\sum_{x\in[0,l]_h}\sum_{\substack{y\in[0,l]_h\\ y\neq 0}} \frac{\left|\varphi^h\left(x+y\right)-\varphi^h\left(x\right)\right|^p}{\left|y\right|^{1+\mu p}} h^2 + \left\|\varphi^h\right\|_{L_{p,h}}^p\right]^{\frac{1}{p}}, 1 \leq p < \infty.$$

This fact follows from the equality $D(A_h^x) = W_{p,h}^2$ for a second order differential operator A_h^x in $L_{p,h}$, $1 < p < \infty$, via the real interpolation method. The alternative method of investigation adopted in (Ashyralyev and Sobolevskii, 1994), (Ashyralyev and Sobolevskii, 2004), based on estimates of fundamental solution of the resolvent equation for the operator A_h^x, allows us to consider also the cases $p = 1$ and $p = \infty$. From that it follows A_h^x was the strongly positive operator in the space $W_{p,h}^{2\alpha}$ for all $0 < 2\alpha < 1, 1 \leq p < \infty$.

In (Ashyralyev and Yaz, 2006), A. Ashyralyev and N. Yaz investigated the differential operator A^x defined by the formula

$$A^x u = -a(x)\frac{d^2u}{dx^2} + \delta u \tag{1.4}$$

with domain

$$D(A^x) = \{u \in C^{(2)}[0,l] : u(0) = u(\mu), u'(0) = u'(l), l/2 \leq \mu \leq l\}. \tag{1.5}$$

Here, $a(x)$ was a smooth function defined on the segment $[0,l]$ and $a(x) \geq a > 0$. They proved that A^x was the strongly positive operator in $C\left[0,l\right]$. For $\alpha \in (0, \frac{1}{2})$, the norms of the space $E_\alpha(C\left[0,l\right], A^x)$ and the Hölder space $C^{2\alpha}\left[0,l\right]$ were equivalent. It follows that A^x was the strongly positive operator in $C^{2\alpha}\left[0,l\right]$.

Ashyralyev A., Nalbant N. and Sozen Y. considered in (Ashyralyev et al., 2014b) the difference operator defined by formula

$$A_h^x u^h = \left\{-a(x_k)\frac{u_{k+1} - 2u_k + u_{k-1}}{h^2} + \delta u_k\right\}_1^{M-1}, \ u^h = \{u_k\}_0^M, Mh = l \tag{1.6}$$

with $u_0 = u_\ell$, $u_1 - u_0 = u_N - u_{N-1}$, where $\ell = \left[\frac{\mu}{h}\right]$, $[\cdot]$ was the greatest integer function. This operator was a first order of approximation of the differential operator A^x defined by formula (1.4) with domain $D(A^x) = \{u \in C^{(2)}[0,l] : u(0) = u(\mu), u'(0) =$

$u'(l), l/2 \leq \mu \leq l\}$. They proved that A_h^x was the strongly positive operator in C_h. For $\alpha \in (0, \frac{1}{2})$, the norms of the space $E_\alpha(C_h, A_h^x)$ and the Hölder space $C_h^{2\alpha}$ were equivalent uniformly in h. It follows that A_h^x was the strongly positive operator in $C_h^{2\alpha}$.

Finally, a survey of results in fractional spaces generated by positive operators and their applications to partial differential equations was given in (Ashyralyev, 2015).

In the present thesis, we will study the positivity of the differential operator A^x defined by the formula

$$A^x u(x) = -u_{xx}(x) + \delta u(x), \delta > 0, 0 < x < 1 \tag{1.7}$$

with domain

$$D(A^x) = \left\{ u \in C^2 [0, 1] : u (0) = 0, u(1) = u (\mu), 0 \leq \mu < 1 \right\},$$

where $\delta > 0$. The structure of fractional spaces generated by this differential operator will be investigated. We will discuss their applications to theory of local and nonlocal boundary value problems for parabolic and elliptic differential equations.

Let us briefly describe the contents of the various chapters of the thesis. It consists of five chapters.

First chapter is the introduction.

Second chapter considers the differential operator A^x defined by formula (1.7).

We will study Green's function of the differential operator A^x defined by formula (1.7). Therefore, we consider the resolvent of the operator $-A^x$, that is, we consider the operator equation

$$A^x u + \lambda u = f \tag{1.8}$$

or

$$\begin{aligned} &\frac{d^2 u(x)}{dx^2} + \delta u(x) + \lambda u(x) = f(x), 0 < x < 1, \\ &u (0) = 0, u(1) = u (\mu), 0 \leq \mu < 1. \end{aligned} \tag{1.9}$$

Pointwise estimates for Green's function of the differential operator A^x defined by formula (1.7) are established.

It will be established the positivity of the differential operator A^x in $C\,[0,1]$. The structure of fractional spaces $E_{\alpha,\infty}(C[0,1],A^x)$ will be investigated. It is established that for any $0<\alpha<1/2$ the norms in the spaces $E_{\alpha,\infty}(C[0,1],A^x)$ and $C^{2\alpha}\,[0,1]$ are equivalent. This result is permitted us to prove the positivity of A^x in $C^{2\alpha}\,[0,1]\,(0<\alpha<1/2)$. In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for elliptic and parabolic equations.

Third chapter establishes the positivity of the differential operator A^x in $L_1\,[0,1]$. The structure of fractional spaces $E_{\alpha,1}(L_1[0,1],A^x)$ will be investigated. It is established that for any $0<\alpha<1/2$ the norms in the spaces $E_{\alpha,1}(L_1[0,1],A^x)$ and $W_1^{2\alpha}\,[0,1]$ are equivalent. This result permits us to prove the positivity of A^x in $W_1^{2\alpha}\,[0,1]\,(0<\alpha<1/2)$. In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for elliptic and parabolic equations.

Fourth chapter establishes the positivity of the differential operator A^x in $L_p\,[0,1]$. The structure of fractional spaces $E_{\alpha,p}(L_p[0,1],A^x)$ will be investigated. It is established that for any $0<\alpha<1/2$ the norms in the spaces $E_{\alpha,p}(L_p[0,1],A^x)$ and $W_p^{2\alpha}\,[0,1]$ are equivalent. This result allows us to prove the positivity of A^x in $W_p^{2\alpha}\,[0,1]\,(0<\alpha<1/2)$. In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for elliptic and parabolic equations.

Fifth chapter contains conclusions.

CHAPTER 2

THE POSITIVITY OF THE SECOND ORDER DIFFERENTIAL OPERATOR WITH THE NONLOCAL CONDITION

In this chapter, we consider the differential operator defined by formula (1.7) .We will study Green's function of this operator. Pointwise estimates for Green's function of the differential operator A^x defined by formula (1.7) are established. The positivity of the differential operator A^x defined by formula (1.7) in $C\,[0,1]$ is established. The structure of fractional spaces $E_{\alpha,\infty}(C[0,1],A^x)$ will be investigated. It is established that for any $0<\alpha<1/2$ the norms in the spaces $E_{\alpha,\infty}(C[0,1],A^x)$ and $C^{2\alpha}\,[0,1]$ are equivalent. This result permits us to prove the positivity of A^x in $C^{2\alpha}\,[0,1]\,(0<\alpha<1/2)$. In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for elliptic and parabolic equations.

2.1 GREEN'S FUNCTION AND POSITIVITY OF A^X DEFINED BY FORMULA (1.7) IN $C\,[0,1]$

First, we will construct Green's function of the differential operator A^x defined by formula (1.7) .

Theorem 2.1.1. *Let $\lambda>0$. Then, the following equation*

$$A^x u+\lambda u=f \tag{2.1}$$

is uniquely solvable, and the following formula holds:

$$u(x) = (A^x + \lambda)^{-1} f(x) = \int_0^1 G(x, s, \mu; \lambda + \delta) f(s) ds, \tag{2.2}$$

where

$$G(x, s, \mu; \lambda + \delta) = -(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}) \frac{T}{2\sqrt{\delta + \lambda}} (e^{-\sqrt{\delta+\lambda}(1-s)} - e^{-\sqrt{\delta+\lambda}(1+s)})$$

$$\times[(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)}) + 1]$$

$$+(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1} \frac{1}{2\sqrt{\delta + \lambda}}$$

$$\times(e^{-\sqrt{\delta+\lambda}|\mu-s|} - e^{-\sqrt{\delta+\lambda}(\mu+s)}) + \frac{1}{2\sqrt{\delta + \lambda}}(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}). \tag{2.3}$$

Here,

$$T = (1 - e^{-2\sqrt{\delta+\lambda}})^{-1}.$$

The function $G(x, s; \lambda + \delta)$ is called Green's function of resolvent equation (2.1).

Proof. We see that problem (1.9) can be obviously rewritten as the equivalent nonlocal boundary value problem for second order linear differential equation

$$-\frac{d^2u}{dx^2} + (\delta + \lambda)u = f(x), 0 < x < 1, u(0) = 0, u(1) = u(\mu), 0 \leq \mu < 1.$$

It is well-known that the following formula

$$u(x) = T\Big\{(e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(2-x)})\varphi + (e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})\psi \tag{2.4}$$

$$-\left(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}\right) \frac{1}{2\sqrt{\delta + \lambda}} \int_0^1 \left(e^{-\sqrt{\delta+\lambda}(1-s)} - e^{-\sqrt{\delta+\lambda}(1+s)}\right) f(s) ds\Big\}$$

$$+\frac{1}{2\sqrt{\delta + \lambda}} \int_0^1 \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right) f(s) ds$$

holds for the solution of the boundary value problem

$$-\frac{d^2u}{dx^2} + (\delta + \lambda)u = f(x), 0 < x < 1, u(0) = \varphi, u(1) = \psi$$

for the second-order linear differential equation. Applying formula (2.4) and local condition $u(0) = \varphi = 0$ and nonlocal boundary condition $u(1) = u(\mu) = \psi$, we get

$$\psi = T\Big\{(e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)})\psi - (e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)})$$

$$\times \frac{1}{2\sqrt{\delta+\lambda}} \int_0^1 \left(e^{-\sqrt{\delta+\lambda}(1-s)} - e^{-\sqrt{\delta+\lambda}(1+s)}\right) f(s)ds \Bigg\}$$

$$+\frac{1}{2\sqrt{\delta+\lambda}} \int_0^1 \left(e^{-\sqrt{\delta+\lambda}|\mu-s|} - e^{-\sqrt{\delta+\lambda}(\mu+s)}\right) f(s)ds.$$

From that it follows that

$$\psi = -(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)}) \tag{2.5}$$

$$\times \frac{1}{2\sqrt{\delta+\lambda}} \int_0^1 \left(e^{-\sqrt{\delta+\lambda}(1-s)} - e^{-\sqrt{\delta+\lambda}(1+s)}\right) f(s)ds$$

$$+(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{T^{-1}}{2\sqrt{\delta+\lambda}} \int_0^1 \left(e^{-\sqrt{\delta+\lambda}|\mu-s|} - e^{-\sqrt{\delta+\lambda}(\mu+s)}\right) f(s)ds.$$

Finally, applying formulas (2.4)-(2.5), we obtain formula (2.2). This finishes the proof of Theorem 2.1.1.

Second, we will study Green's function of the differential operator A^x defined by formula (1.7) .

Lemma 2.1.2. *For all* $0 \le x \le 1$, *the following formula holds*

$$\int_0^1 G(x,s,\mu;\lambda+\delta)ds = \frac{1}{\delta+\lambda} - \frac{1}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x} \tag{2.6}$$

$$-\frac{e^{-\sqrt{\delta+\lambda}\mu}}{\delta+\lambda}\left(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}\right)\left(1+e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)^{-1}.$$

Proof Applying formula (2.3) and taking the integral, we get

$$\int_0^1 G(x,s,\mu;\lambda+\delta)ds = -\int_0^1 \left(e^{-\sqrt{\delta+\lambda}(1-s)} - e^{-\sqrt{\delta+\lambda}(1+s)}\right) ds$$

$$\times \left(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}\right)\frac{T}{2\sqrt{\delta+\lambda}}$$

$$\times\left[\left(1-e^{-\sqrt{\delta+\lambda}(1-\mu)}\right)^{-1}\left(1+e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)^{-1}\left(e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)+1\right]$$

$$+\left(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}\right)\left(1-e^{-\sqrt{\delta+\lambda}(1-\mu)}\right)^{-1}\left(1+e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)^{-1}\frac{1}{2\sqrt{\delta+\lambda}}$$

$$\times \int_0^1 \left(e^{-\sqrt{\delta+\lambda}|\mu-s|} - e^{-\sqrt{\delta+\lambda}(\mu+s)}\right) ds + \frac{1}{2\sqrt{\delta+\lambda}} \int_0^1 \left(e^{-\sqrt{\delta+\lambda}|x-s|} - e^{-\sqrt{\delta+\lambda}(x+s)}\right) ds$$

$$= -\left(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)}\right) \frac{T}{2(\delta+\lambda)} \left(1 - 2e^{-\sqrt{\delta+\lambda}} + e^{-2\sqrt{\delta+\lambda}}\right)$$

$$\times \left[\left(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)}\right)^{-1} \left(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)^{-1} \left(e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)}\right) + 1\right]$$

$$+(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}$$

$$\times \frac{1}{2(\delta+\lambda)} \left(2 - 2e^{-\sqrt{\delta+\lambda}\mu} - e^{-\sqrt{\delta+\lambda}(1-\mu)} + e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)$$

$$+\frac{1}{2(\delta+\lambda)} \left(2 - 2e^{-\sqrt{\delta+\lambda}x} - e^{-\sqrt{\delta+\lambda}(1-x)} + e^{-\sqrt{\delta+\lambda}(1+x)}\right)$$

$$= \frac{1}{\delta+\lambda} - \frac{1}{\delta+\lambda} e^{-\sqrt{\delta+\lambda}x} + \frac{1}{2(\delta+\lambda)} (e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})$$

$$\times (1 - e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Delta,$$

where

$$\Delta = -\left(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)}\right)\left(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)}\right) - T\left(1 - e^{-\sqrt{\delta+\lambda}}\right)^2$$

$$\times \left[e^{-\sqrt{\delta+\lambda}(1-\mu)} - e^{-\sqrt{\delta+\lambda}(1+\mu)} + \left(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)}\right)\left(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)}\right)\right]$$

$$+2 - 2e^{-\sqrt{\delta+\lambda}\mu} - e^{-\sqrt{\delta+\lambda}(1-\mu)} + e^{-\sqrt{\delta+\lambda}(1+\mu)}$$

$$= -2e^{-\sqrt{\delta+\lambda}\mu}\left(1 - e^{-\sqrt{\delta+\lambda}(1-\mu)}\right).$$

This finishes the proof of Lemma 2.1.2.

Lemma 2.1.3. *For all* $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| \le \varphi, \varphi < \pi/2\}$, *expressions* $1 + e^{-\sqrt{\delta+\lambda}(1+\mu)}$, $1 - e^{-\sqrt{\delta+\lambda}(1-\mu)}$ *and* $1 - e^{-2\sqrt{\delta+\lambda}}$ *are not equal to zero.*

Proof Let $\lambda = \rho e^{i\varphi} = \rho\cos\varphi + i\sin\varphi$. Then,

$$\delta + \lambda = \delta + \rho\cos\varphi + i\rho\sin\varphi = |\delta+\lambda|\, e^{i\psi}$$

and

$$|\delta+\lambda| = \sqrt{\delta^2 + 2\rho\cos\varphi + \rho^2} \ge \sqrt{\delta^2 + |\lambda|^2}.$$

Therefore, $(\delta+\lambda)^{1/2} = \left|(\delta+\lambda)^{1/2}\right| e^{i\psi/2}, \left|(\delta+\lambda)^{1/2}\right| = |\delta+\lambda|^{1/2}\frac{\sqrt{2}}{2}$. Here, $\tan\psi = \frac{\rho\sin\varphi}{\delta+\rho\cos\varphi} \le \tan\varphi$. From that it follows $|\delta+\lambda|^{1/2} \ge \left(\delta^2 + |\lambda|^2\right)^{1/4}$and we have that $\left|(\delta+\lambda)^{1/2}\right| \ge \left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}$.Thus, using the triangle inequality, we obtain

$$\left|1 - e^{-2\sqrt{\delta+\lambda}}\right| \ge 1 - \left|e^{-2\sqrt{\delta+\lambda}}\right| \ge 1 - e^{-2\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}} > 0.$$

Similarly, $\left|1+e^{-\sqrt{\delta+\lambda}(1+\mu)}\right|>0$ and $\left|1-e^{-\sqrt{\delta+\lambda}(1-\mu)}\right|>0$. Lemma 2.1.3 is proved.

Lemma 2.1.4. *For any* $\lambda \in R_{\varphi}=\{\lambda : |\arg\lambda| \le \varphi, \varphi<\pi/2\}$, $\mu \in [0,1)$ *and* $x \in [0,1]$, *the following pointwise estimates hold*

$$|G(x,s,\mu;\lambda+\delta)| \le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \tag{2.7}$$

for $0 \le x \le \frac{1+\mu}{2}$ *and*

$$|G(x,s,\mu;\lambda+\delta)|$$

$$\le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \begin{cases} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}, 0 \le s \le \mu, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)},\ \mu \le s \le x+\frac{\mu-1}{2}, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|},\ x+\frac{\mu-1}{2} \le s \le 1 \end{cases} \tag{2.8}$$

for $\frac{1+\mu}{2} \le x \le 1$.

Proof. Let $A=\min\{2-x-s, 1-x+|\mu-s|, |x-s|\}$. First, we prove that

$$2-x-s \ge |x-s| \tag{2.9}$$

for any $x,s \in [0,1]$. Actually, we have that $2s \le 2$. By using properties of inequality, we get

$$s-x \le 2-s-x. \tag{2.10}$$

Since $2x \le 2$, it follows

$$x-s \le 2-x-s. \tag{2.11}$$

From estimates (2.10) and (2.11), it follows (2.9).

Applying (2.9), we obtain that $\min\{2-x-s, |x-s|\}=|x-s|$ and $A=\min\{1-x+|\mu-s|, |x-s|\}$.

Second, we obtain A. We consider two cases: $0 \le x \le \frac{1+\mu}{2}$ and $\frac{1+\mu}{2} \le x \le 1$.

Let $0 \le x \le \frac{1+\mu}{2}$ for any $s \in [0,1]$.

Assume that $0 \le s \le \mu$. Then, $|\mu-s|=\mu-s$ and $A=\min\{1-x+\mu-s, |x-s|\}$.

Since $x \le \frac{1+\mu}{2}$, we have that

$$x-s \le 1-x+\mu-s. \tag{2.12}$$

By adding inequalities $s \leq 1$ and $s \leq \mu$ side by side, we get $2s \leq 1 + \mu$. Then,

$$s - x \leq 1 - x + \mu - s. \tag{2.13}$$

From estimates (2.12) and (2.13) it follows

$$|x - s| \leq 1 - x + \mu - s. \tag{2.14}$$

Assume that $\mu \leq s \leq 1$. Then, $|\mu - s| = s - \mu$ and $A = \min\{1 - x + s - \mu, |x - s|\}$. Since $\mu \leq 1$, adding $s - x$ both side, we get

$$\mu + s - x \leq 1 + s - x$$

or

$$|x - s| = s - x \leq 1 - x + s - \mu.$$

For $s \geq x$, we have that $|x - s| = s - x$ and

$$|x - s| \leq 1 - x + s - \mu. \tag{2.15}$$

Let $s \leq x$, then $|x - s| = x - s$. By adding inequalities $\mu \leq s$ and $x \leq \frac{1+\mu}{2}$ side by side, we obtain

$$x + \mu \leq \frac{1 + \mu}{2} + s.$$

Therefore,

$$2x - 2s \leq 1 + \mu - 2\mu$$

or

$$x - s \leq 1 - x + s - \mu. \tag{2.16}$$

From estimates (2.15) and (2.16), it follows

$$|x - s| \leq 1 - x + \mu - s. \tag{2.17}$$

Using estimates (2.14) and (2.17), we get

$$A = \min\{1 - x + |\mu - s|\,, |x - s|\} = |x - s| \tag{2.18}$$

for any $0 \le x \le \frac{1+\mu}{2}$ and $s \in [0,1]$. By Lemma 2.1.3 and estimate (2.18), we have the following estimate

$$|G(x,s,\mu;\lambda+\delta)| \le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \tag{2.19}$$

for any $0 \le x \le \frac{1+\mu}{2}$ and $s \in [0,1]$.

Let $\frac{1+\mu}{2} \le x \le 1$ for any $s \in [0,1]$. Suppose that $0 \le s \le \mu$. Then, $|x-s| = x-s$ and we have

$$2x \ge 1+\mu,$$

$$x-s \ge 1+\mu-x-s.$$

Then,

$$A = 1-x+\mu-s. \tag{2.20}$$

Assume that $\mu \le s \le \frac{1+\mu}{2}$. Then, $|x-s| = x-s$. Since

$$1-x+s-\mu = x-s,$$

$$2s = 2x+\mu-1,$$
$$s = x+\frac{\mu-1}{2},$$

we have that

$$1-x+s-\mu \le x-s = |x-s|$$

for $\mu \le s \le x+\frac{\mu-1}{2}$ and

$$1-x+s-\mu \ge x-s = |x-s|$$

for $x+\frac{\mu-1}{2} \le s \le \frac{1+\mu}{2}$. Therefore,

$$A = \begin{cases} 1-x+s-\mu, & \mu \le s \le x+\frac{\mu-1}{2}, \\ |x-s|, & x+\frac{\mu-1}{2} \le s \le \frac{1+\mu}{2}. \end{cases} \tag{2.21}$$

Assume that $\frac{1+\mu}{2} \le s \le 1$. For $x > s$, we have that

$$|x-s| = x-s, |\mu-s| = s-\mu.$$

Applying inequalities $x \le 1$ and $\frac{1+\mu}{2} \le s$, we get

$$x+\frac{1+\mu}{2} \le 1+s.$$

Then,

$$2x + 1 + \mu \le 2 + 2s$$

or

$$|x - s| = x - s \le 1 - x + s - \mu. \tag{2.22}$$

For $x < s$, we have that

$$|x - s| = s - x, |\mu - s| = s - \mu.$$

Applying $\mu \le 1$, we obtain

$$\mu + s - x \le 1 + s - x$$

or

$$|x - s| = s - x \le 1 - x + s - \mu. \tag{2.23}$$

From estimates (2.22) and (2.23) it follows

$$|x - s| \le 1 - x + s - \mu. \tag{2.24}$$

By using (2.20), (2.21), and (2.24), we get

$$A = \begin{cases} 1 - x + \mu - s, \ 0 \le s \le \mu, \\ \\ 1 - x + s - \mu, \ \mu \le s \le x + \frac{\mu - 1}{2}, \\ \\ |x - s| \ , \ x + \frac{\mu - 1}{2} \le s \le 1. \end{cases} \tag{2.25}$$

for any $\frac{1+\mu}{2} \le x \le 1$ and $s \in [0, 1]$. Lemma 2.1.3 and formula (2.25) yields the following estimate

$$|G(x, s, \mu; \lambda + \delta)| \le \frac{M_1(\delta, \mu)}{\left(\delta^2 + |\lambda|^2\right)^{1/4}}$$

$$\times \begin{cases} e^{-\left(\delta^2 + |\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(1 - x + \mu - s)}, 0 \le s \le \mu, \\ \\ e^{-\left(\delta^2 + |\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(1 - x + s - \mu)}, \ \mu \le s \le x + \frac{\mu - 1}{2}, \\ \\ e^{-\left(\delta^2 + |\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|x - s|}, \ x + \frac{\mu - 1}{2} \le s \le 1 \end{cases}$$

for any $\frac{1+\mu}{2} \le x \le 1$ and $s \in [0, 1]$. This finishes the proof of Lemma 2.1.4.

Lemma 2.1.5. *For any* $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| \leq \varphi, \varphi < \pi/2\}$, $\mu \in [0,1)$ *and* $x \in [0,1]$, *the following estimate*

$$\int_0^1 |G(x,s,\mu;\lambda+\delta)|\, ds \leq \frac{M(\delta,\mu)}{1+|\lambda|} \tag{2.26}$$

is valid.

Proof. Let $0 \leq x \leq \frac{1+\mu}{2}$ for any $s \in [0,1]$. Then, applying (2.19), we get

$$\int_0^1 |G(x,s,\mu;\lambda+\delta)|\, ds \leq \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} ds$$

$$= \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left[\int_0^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)} ds + \int_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)} ds\right]$$

$$\leq \frac{M_2(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/2}}. \tag{2.27}$$

Estimate (2.26) for this case follows from the last estimate and the following inequality

$$\frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/2}} \leq \frac{M_1(\delta)}{1+|\lambda|}. \tag{2.28}$$

Let $\frac{1+\mu}{2} \leq x \leq 1$ for any $s \in [0,1]$. Then, from (2.8) it follows that

$$\int_0^1 |G(x,s,\mu;\lambda+\delta)|\, ds \leq \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left[\int_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)} ds\right.$$

$$\left. + \int_\mu^{x+\frac{\mu-1}{2}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)} ds + \int_{x+\frac{\mu-1}{2}}^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} ds\right]$$

$$\leq \frac{M_3(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/2}}. \tag{2.29}$$

Estimate (2.30) for this case follows from estimate (2.29) and inequality (2.28). This ends the proof of Lemma 2.1.5.

Lemma 2.1.6. *For any* $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| \leq \varphi, \varphi < \pi/2\}$, $\mu \in [0,1)$ *and* $x \in [0,1]$, *the following estimate for the derivative of Green's function of resolvent equation (2.1) with respect to x holds*

$$|G_x(x,s,\mu;\lambda+\delta)| \leq M_2(\delta,\mu)e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \tag{2.30}$$

for $0 \leq x \leq \frac{1+\mu}{2}$,

$$|G_x(x,s,\mu;\lambda+\delta)| \leq M_3(\delta,\mu)$$
$$\times\begin{cases} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}, 0 \leq s \leq \mu, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)},\ \mu \leq s \leq x+\frac{\mu-1}{2}, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|},\ x+\frac{\mu-1}{2} \leq s \leq 1 \end{cases} \tag{2.31}$$

for $\frac{1+\mu}{2} < x \leq 1$.

Proof. Using equation (2.3), we get

$$\begin{aligned} G_x(x,s,\mu;\lambda+\delta) &= -\frac{T}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(e^{-\sqrt{\delta+\lambda}(1-s)}-e^{-\sqrt{\delta+\lambda}(1+s)}) \\ &\times[(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)}-e^{-\sqrt{\delta+\lambda}(1+\mu)})+1] \\ &+\frac{1}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1} \\ &\times(e^{-\sqrt{\delta+\lambda}|\mu-s|}-e^{-\sqrt{\delta+\lambda}(\mu+s)})+\frac{1}{2}(e^{-\sqrt{\delta+\lambda}(s-x)}+e^{-\sqrt{\delta+\lambda}(x+s)}) \end{aligned} \tag{2.32}$$

for $x-s<0$. If $x-s>0$, then using equation (2.3), we get

$$\begin{aligned} G_x(x,s,\mu;\lambda+\delta) &= -\frac{T}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(e^{-\sqrt{\delta+\lambda}(1-s)}-e^{-\sqrt{\delta+\lambda}(1+s)}) \\ &\times[(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}(e^{-\sqrt{\delta+\lambda}(1-\mu)}-e^{-\sqrt{\delta+\lambda}(1+\mu)})+1] \\ &+\frac{1}{2}(e^{-\sqrt{\delta+\lambda}(1-x)}+e^{-\sqrt{\delta+\lambda}(1+x)})(1-e^{-\sqrt{\delta+\lambda}(1-\mu)})^{-1}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1} \\ &\times(e^{-\sqrt{\delta+\lambda}|\mu-s|}-e^{-\sqrt{\delta+\lambda}(\mu+s)})+\frac{1}{2}(-e^{-\sqrt{\delta+\lambda}(x-s)}+e^{-\sqrt{\delta+\lambda}(x+s)}). \end{aligned} \tag{2.33}$$

There are two possible cases: $0 \leq x \leq \frac{1+\mu}{2}$ and $\frac{1+\mu}{2} < x \leq 1$. In the first case, we will estimate (2.18). By Lemma 2.1.3 and estimate (2.18), we have the following estimate from (2.32) and (2.33).

$$|G_x(x,s,\mu;\lambda+\delta)| \leq M_1(\delta,\mu)e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}. \tag{2.34}$$

In the second case, from (2.32), by Lemma 2.1.3 and formula (2.25) we have the following estimate

$$|G_x(x,s,\mu;\lambda+\delta)|$$

$$\leq M_1(\delta,\mu)\begin{cases} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}, 0\leq s\leq\mu, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)},\ \mu\leq s\leq x+\frac{\mu-1}{2}, \\ e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|},\ x+\frac{\mu-1}{2}\leq s\leq 1. \end{cases} \tag{2.35}$$

Lemma 2.1.6 is proved.

Third, the positivity of A^x in $C\,[0,1]$ is investigated.

Theorem 2.1.7. *For all $\lambda\in R_\varphi=\{\lambda:|\arg\lambda|\leq\varphi,\varphi<\pi/2\}$, the resolvent $(\lambda I+A^x)^{-1}$ defined by formula (2.4) is subject to the bound*

$$\left\|(\lambda I+A^x)^{-1}\right\|_{C[0,1]\to C[0,1]}=\frac{M(\varphi,\delta)}{1+|\lambda|}.$$

Proof. *Using formula (2.4) and the triangle inequality, we get*

$$|u(x)|\leq\int\limits_0^1|G(x,s,\mu;\lambda+\delta)|\,ds\max_{0\leq s\leq 1}|f(s)|$$

for any $x\in[0,1]$. So, we get

$$\max_{x\in[0,1]}|u(x)|\leq\max_{x\in[0,1]}\int\limits_0^1|G(x,s,\mu;\lambda+\delta)|\,ds\,\|f\|_{C[0,1]}\,.$$

Then, we have

$$\left\|(A^x+\lambda)^{-1}f\right\|_{C[0,1]}\leq\frac{M(\varphi,\delta)}{1+|\lambda|}\,\|f\|_{C[0,1]}\,.$$

From that we obtain

$$\left\|(A^x+\lambda)^{-1}\right\|_{C[0,1]\to C[0,1]}\leq\frac{M(\varphi,\delta)}{1+|\lambda|}.$$

This is the end of Theorem 2.1.7.

Clearly, the operators A^x and its resolvent $(\lambda + A^x)^{-1}$ commute. Thus, from the definition of the norm in the space $E_\alpha(C[0,1], A^x)$ it follows that

$$\left\|(\lambda + A^x)^{-1}\right\|_{E_\alpha(C[0,1],A^x)\to E_\alpha(C[0,1],A^x)} \leq \left\|(\lambda + A^x)^{-1}\right\|_{C[0,1]\to C[0,1]}.$$

Hence, by using Theorem 2.1.7, we obtain the positivity of the operator A^x in the fractional spaces $E_\alpha(C[0,1], A^x)$.

2.2 THE STRUCTURE OF FRACTIONAL SPACES $E_\alpha(C[0,1], A^X)$, POSITIVITY OF A^X IN $\overset{\circ}{C}{}^{2\alpha}[0,1]$

Now, we will study the positivity of A^x in $\overset{\circ}{C}{}^{2\alpha}[0,1]$. We have the following theorem.

Theorem 2.2.1. *Let* $\alpha \in (0, 1/2)$. *Then, the norms of the spaces* $E_\alpha(C[0,1], A^x)$ *and* $\overset{\circ}{C}{}^{2\alpha}[0,1]$ *are equivalent.*

Proof. For any $\lambda \geq 0$, we have the following equality

$$A^x(\lambda + A^x)^{-1}f(x) = f(x) - \lambda(\lambda + A^x)^{-1}f(x).$$

By formula (2.2), we can write

$$A^x(\lambda + A^x)^{-1}f(x) = f(x) - \lambda \int_0^1 G(x, s, \mu; \lambda + \delta)f(s)ds$$

$$= \frac{\delta}{\delta + \lambda}f(x) + \frac{\lambda}{\delta + \lambda}f(x) - \lambda \int_0^1 G(x, s, \mu; \lambda + \delta)f(s)ds. \tag{2.36}$$

From equation (2.6) it follows the following formula

$$\frac{1}{\delta + \lambda} = \int_0^1 G(x, s, \mu; \lambda + \delta)ds + \frac{1}{\delta + \lambda}e^{-\sqrt{\delta+\lambda}x}$$

$$+\frac{e^{-\sqrt{\delta+\lambda}\mu}}{\delta + \lambda}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}. \tag{2.37}$$

Using formulas (2.36) and (2.37), we get

$$A^x(\lambda+A^x)^{-1}f(x)=\frac{\delta}{\delta+\lambda}f(x)+\frac{\lambda}{\delta+\lambda}\left[e^{-\sqrt{\delta+\lambda}x}\right. \tag{2.38}$$

$$\left.+e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\right]f(x)$$

$$+\lambda\int_0^1 G(x,s,\mu;\lambda+\delta)\,(f(x)-f(s))ds.$$

Then,

$$\lambda^\alpha A^x(\lambda+A^x)^{-1}f(x)=\frac{\delta\lambda^\alpha}{\delta+\lambda}f(x)+\frac{\lambda^{\alpha+1}}{\delta+\lambda}\left[e^{-\sqrt{\delta+\lambda}x}\right.$$

$$\left.+e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\right]f(x)$$

$$+\lambda^{\alpha+1}\int_0^1 G(x,s,\mu;\lambda+\delta)\,(f(x)-f(s))ds$$

$$=P_1(x)+P_2(x)+P_3(x),$$

where

$$P_1(x)=\frac{\delta\lambda^\alpha}{\delta+\lambda}f(x),$$

$$P_2(x)=\frac{\lambda^{\alpha+1}}{\delta+\lambda}\left[e^{-\sqrt{\delta+\lambda}x}+e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})\right.$$

$$\left.\times(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\right]f(x),$$

$$P_3(x)=\lambda^{\alpha+1}\int_0^1 G(x,s,\mu;\lambda+\delta)\,(f(x)-f(s))ds.$$

Using the definition of norm space $\overset{\circ}{C}{}^{2\alpha}[0,1]$ and $\frac{\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda}\leq 1$, we can write

$$|P_1(x)|\leq\frac{\delta^\alpha\lambda^\alpha\delta^{1-\alpha}}{\delta+\lambda}|f(x)|\leq\delta^\alpha\max_{x\in[0,1]}|f(x)|\leq\delta^\alpha\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}$$

for any $x\in[0,1]$. Then,

$$\max_{x\in[0,1]}|P_1(x)|\leq\delta^\alpha\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}$$

or

$$\|P_1\|_{C[0,1]}\leq\delta^\alpha\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}. \tag{2.39}$$

We have that

$$P_2(x)=\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x}\,[f(x)-f(0)]$$

$$+\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu-x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\left[f(x)-f(1)\right]$$

$$+\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\left[-f(x)+f(1)\right]$$

$$+\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu-x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\left[f(\mu)-f(0)\right]$$

$$-\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}f(1).$$

Then, using the triangle inequality and the estimate $p^{\alpha}e^{-pt}\leq\frac{M}{t^{\alpha}}$, $p>0$, we can write

$$|P_2(x)|\leq\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}x}\frac{|f(x)-f(0)|}{x^{2\alpha}}x^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu-x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{|f(x)-f(1)|}{(1-x)^{2\alpha}}(1-x)^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{|f(x)-f(1)|}{(1-x)^{2\alpha}}(1-x)^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\frac{|f(\mu)-f(0)|}{\mu^{2\alpha}}\mu^{2\alpha}$$

$$+\left(\frac{\lambda}{\delta+\lambda}\right)^{\alpha+1}(\delta+\lambda)^{\alpha+1}e^{-\sqrt{\delta+\lambda}(1+\mu+x)}(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}f(1)$$

$$\leq M\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}+M\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}+M\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}+M\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}$$

$$+\max_{x\in[0,1]}|f(x)|\frac{1}{(1+\mu+x)^{2\alpha}}\leq M_1\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}$$

for any $x\in[0,1]$. Thus,

$$\|P_2\|_{C[0,1]}\leq M\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}. \qquad (2.40)$$

Now, we will estimate $P_3(x)$. From the estimate (2.19) for $0\leq x\leq\mu$ and $\mu<x\leq\frac{1+\mu}{2}$, we get

$$|P_3(x)|\leq\lambda^{\alpha+1}\int_0^1|G(x,s,\mu;\lambda+\delta)|\,|f(x)-f(s)|\,ds$$

$$\leq\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\,|f(x)-f(s)|\,ds$$

$$\leq\frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\,|f(x)-f(s)|\,\frac{|x-s|^{2\alpha}}{|x-s|^{2\alpha}}ds$$

$$\leq M \|f\|_{C^{2\alpha}[0,1]} \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int\limits_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} |x-s|^{2\alpha} ds.$$

Using the substitution $y=\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|$, we obtain

$$|P_3(x)| \leq M_2 \|f\|_{\overset{\circ}{C}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \int\limits_0^\infty e^{-y} y^{2\alpha} ds$$

$$\leq M_2 \|f\|_{\overset{\circ}{C}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}\Gamma(2\alpha+1)}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \leq M_2 \|f\|_{\overset{\circ}{C}^{2\alpha}[0,1]} \Gamma(2\alpha+1).$$

Using estimate (2.8) for $\frac{1+\mu}{2}<x\leq 1$, we get

$$P_3(x) \leq \lambda^{\alpha+1} \int\limits_0^1 G(x,s,\mu;\lambda+\delta)\left(f(x)-f(s)\right)ds = P_{31}(x)+P_{32}(x)+P_{32}(x),$$

where

$$P_{31}(x) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)} \left(f(x)-f(s)\right)ds,$$

$$P_{32}(x) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int\limits_\mu^x e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)} \left(f(x)-f(s)\right)ds,$$

$$P_{33}(x) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int\limits_x^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)} \left(f(x)-f(s)\right)ds.$$

Let us estimate $P_{31}(x)$.

$$|P_{31}(x)| = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)} \left(|f(x)-f(1)+f(\mu)-f(s)|\right)ds.$$

By triangle inequality, we have

$$|P_{31}(x)| \leq \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)} \left[|f(x)-f(1)|+|f(\mu)-f(s)|\right]ds.$$

Using the definition of norm, the following inequality holds

$$|P_{31}(x)| \leq \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \|f\|_{\overset{\circ}{C}^{2\alpha}[0,1]} \left[\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}(1-x)^{2\alpha}ds\right.$$

$$\left.+\int\limits_0^\mu e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}(\mu-s)^{2\alpha}ds\right]$$

$$\leq \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}\, e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)}(1-x)^{2\alpha}\left[\int\limits_0^{\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(\mu-s)}ds\right.$$

$$\left.+e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)}\int\limits_0^{\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(\mu-s)}(\mu-s)^{2\alpha}ds\right].$$

Let

$$I_1 = e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)}(1-x)^{2\alpha},$$

$$I_2 = \int\limits_0^{\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(\mu-s)}ds,$$

$$I_3 = \int\limits_0^{\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}(\mu-s)^{2\alpha}ds.$$

Hence, by the inequality $e^{-at} \leq \frac{M}{(at)^{\theta}}; 0<\theta<1$, we get

$$I_1 \leq \frac{(1-x)^{2\alpha}}{\left(\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)\right)^{2\alpha}} = \frac{M}{\left(\delta^2+|\lambda|^2\right)^{2\alpha/4}}.$$

Taking the integral, we have

$$I_2 \leq \frac{\left[1-e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}\mu}\right]}{\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}} \leq \frac{M}{\left(\delta^2+|\lambda|^2\right)^{1/4}},$$

and for $y = (\delta^2+|\lambda|^{21/4}\frac{\sqrt{2}}{2}(\mu-s)$

$$I_3 \leq \frac{M}{\left(\delta^2+|\lambda|^2\right)^{\frac{2\alpha+1}{4}}}\int\limits_0^{\infty} e^{-y}y^{2\alpha}ds \leq \frac{M\Gamma\left(2\alpha+1\right)}{\left(\delta^2+|\lambda|^2\right)^{\frac{2\alpha+1}{4}}} \leq \frac{M}{\left(\delta^2+|\lambda|^2\right)^{\frac{2\alpha+1}{4}}}.$$

Consequently, if we say $e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x)} \leq 1$, $\Gamma\left(2\alpha+1\right) \leq 1$ and using the estimates for I_1, I_2,I_3 the following inequality holds

$$|P_{31}(x)| \leq \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}\left[\frac{M}{\left(\delta^2+|\lambda|^2\right)^{2\alpha/4}}\cdot\frac{M}{\left(\delta^2+|\lambda|^2\right)^{1/4}}+\frac{M}{\left(\delta^2+|\lambda|^2\right)^{\frac{2\alpha+1}{4}}}\right],$$

$$|P_{31}(x)| \leq M_2(\delta,\mu)\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}\frac{\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \leq M_2(\delta,\mu)\|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}.$$

Let us estimate $P_{32}(x)$.

$$P_{32}(x) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\mu}^{x} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)}\left(f(x)-f(s)\right)ds,$$

$$|P_{32}(x)| \leq M \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{\mu}^{x} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(x-s)} (x-s)^{2\alpha}\, ds.$$

Using the substitution $y = \left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(x-s)$, we obtain the following inequality

$$|P_{32}(x)| \leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \left(-\int_{\infty}^{0} e^{-y} y^{2\alpha} ds\right)$$

$$\leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \int_{0}^{\infty} e^{-y} y^{2\alpha} ds$$

$$\leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}\Gamma(2\alpha+1)}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \leq M_3 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \Gamma(2\alpha+1).$$

Let us estimate $P_{33}(x)$.

$$P_{33}(x) = \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{x}^{1} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)} (f(x)-f(s))\, ds.$$

So,

$$|P_{33}(x)| \leq M \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{M_1(\delta,\mu)\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{x}^{1} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(s-x)} |x-s|^{2\alpha}\, ds.$$

Using the substitution $y = \left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(s-x)$, we have the following inequality

$$|P_{33}(x)| \leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \int_{0}^{\infty} e^{-y} y^{2\alpha} ds$$

$$\leq M_2 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \frac{\lambda^{\alpha+1}\Gamma(2\alpha+1)}{\left(\delta^2+|\lambda|^2\right)^{\frac{1+\alpha}{2}}} \leq M_3 \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \Gamma(2\alpha+1).$$

Then, we can write the following

$$\max_{x\in[0,1]} |P_3(x)| \leq \max_{x\in[0,1]} |P_{31}(x)| + \max_{x\in[0,1]} |P_{32}(x)| + \max_{x\in[0,1]} |P_{33}(x)|$$

or

$$\|P_3\|_{C[0,1]} \leq M(\alpha) \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}. \tag{2.41}$$

Using estimates (2.37),(2.39), and (2.40), we get

$$\max_{x\in[0,1]} \left|\lambda^{\alpha} A^x (\lambda + A^x)^{-1} f(x)\right| \leq M(\delta,\mu) \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} + M(\alpha) \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]} \tag{2.42}$$

for any $\lambda \geq 0$. Hence,

$$\|f\|_{E_\alpha(C[0,1],A^x)} \leq M \|f\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]}.$$

Now, let us prove the reverse inequality. For any positive operator A^x in the Banach space, we can write

$$I = \int_0^\infty A^x(\lambda + A^x)^{-2} d\lambda, \tag{2.43}$$

where I is the identity operator.

From formulas (2.2) and (2.43) it follows that

$$f(x) = \int_0^\infty (\lambda + A^x)^{-1} A^x (\lambda + A^x)^{-1} f(x) d\lambda$$

$$= \int_0^\infty \int_0^1 G(x, s, \mu; \lambda + \delta) A^s (\lambda + A^s)^{-1} f(s) ds d\lambda.$$

Consequently,

$$f(x+\tau) - f(x) = \int_0^\infty \int_0^1 \left[G(x+\tau, s, \mu; \lambda + \delta) - G(x, s, \mu; \lambda + \delta)\right] A^s(\lambda + A^s)^{-1} f(s) ds d\lambda$$

$$= \int_0^\infty \lambda^{-\alpha} \int_0^1 \left[G(x+\tau, s, \mu; \lambda + \delta) - G(x, s, \mu; \lambda + \delta)\right] \lambda^\alpha A^s(\lambda + A^s)^{-1} f(s) ds d\lambda. \tag{2.44}$$

Hence,

$$|f(x+\tau) - f(x)| \leq \left(\int_0^\infty \lambda^{-\alpha} \int_0^1 |G(x+\tau, s, \mu; \lambda + \delta) - G(x, s, \mu; \lambda + \delta)| \, ds d\lambda \right)$$
$$\times \|f\|_{E_\alpha(C[0,1],A^x)}.$$

Let

$$T = \tau^{-2\alpha} \left(\int_0^\infty \lambda^{-\alpha} \int_0^1 |G(x+\tau, s, \mu; \lambda + \delta) - G(x, s, \mu; \lambda + \delta)| \, ds d\lambda \right)$$

$$= \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_0^x |G(x+\tau, s, \mu; \lambda + \delta) - G(x, s, \mu; \lambda + \delta)| \, ds d\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_x^{x+\tau} |G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)|\, dsd\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_{x+\tau}^{1} |G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)|\, dsd\lambda$$

$$=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_0^{x}\int_x^{x+\tau} |G_z(z,,s,\mu;\lambda+\delta)|\, dzdsd\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_x^{x+\tau} |G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)|\, dsd\lambda$$

$$+\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_{x+\tau}^{1}\int_x^{x+\tau} |G_z(z,,s,\mu;\lambda+\delta)|\, dzdsd\lambda$$

$$=T_1+T_2+T_3.$$

Here,

$$T_1=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_0^{x}\int_x^{x+\tau} |G_z(z,,s,\mu;\lambda+\delta)|\, dzdsd\lambda,$$

$$T_2=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_x^{x+\tau} |G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)|\, dsd\lambda,$$

$$T_3=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_{x+\tau}^{1}\int_x^{x+\tau} |G_z(z,,s,\mu;\lambda+\delta)|\, dzdsd\lambda.$$

Then, for any $x,\tau\in R^+$, we have that

$$\frac{|f(x+\tau)-f(x)|}{|\tau|^{-2\alpha}}\le T\,\|f\|_{E_\alpha(C^{2\alpha}[0,1],A^x).}$$

Now, we will prove that

$$T\le\frac{M(\delta)}{2\alpha\,(1-2\alpha)}. \tag{2.45}$$

We will estimate T_1,T_2 and T_3. First, let us estimate T_1.

$$T_1=\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_0^{x}\int_x^{x+\tau} |G_z(z,,s,\mu;\lambda+\delta)|\, dzdsd\lambda$$

$$\le\tau^{-2\alpha}\int_0^\infty \lambda^{-\alpha}\int_0^{x}\int_x^{x+\tau} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|z-s|}dzdsd\lambda$$

$$\leq \tau^{-2\alpha} \int_0^x \int_x^{x+\tau} \int_0^\infty \lambda^{-\alpha} e^{-|\lambda|^{1/2} \frac{\sqrt{2}}{2}|z-s|} d\lambda dz ds,$$

where

$$\Delta(z,s) = \int_0^\infty \lambda^{-\alpha} e^{-|\lambda|^{1/2} \frac{\sqrt{2}}{2}|z-s|} d\lambda. \tag{2.46}$$

By changing variable

$$p = -|\lambda|^{1/2} \frac{\sqrt{2}}{2} |z-s|,$$

we get

$$\lambda = \left(\frac{p}{\frac{\sqrt{2}}{2}|z-s|} \right)^2 = \frac{p^2}{\frac{1}{2}|z-s|^2}, d\lambda = \frac{2pdp}{\frac{1}{2}|z-s|^2}.$$

Then,

$$\Delta = \int_0^\infty \frac{|z-s|^{2\alpha} \frac{1}{2\alpha}}{p^{2\alpha}} e^{-p} \frac{2pdp}{\frac{1}{2}|z-s|^2} = |z-s|^{2\alpha-2} 2^{2-\alpha} \int_0^\infty e^{-p} p^{1-2\alpha} dp$$

$$= |z-s|^{2\alpha-2} 2^{2-\alpha} \Gamma(2-2\alpha) = M_0 |z-s|^{2\alpha-2}.$$

From that, it follows

$$T_1 \leq M_1 \tau^{-2\alpha} \int_0^x \int_x^{x+\tau} (z-s)^{2\alpha-2} dz ds$$

$$\leq M_2 \tau^{-2\alpha} \int_x^{x+\tau} \frac{(z-s)^{2\alpha-1}}{1-2\alpha} dz$$

$$= M_2 \frac{\tau^{-2\alpha}\tau^{2\alpha}}{(1-2\alpha)2\alpha} = \frac{M_2}{(1-2\alpha)2\alpha}. \tag{2.47}$$

Second, let us estimate T_2.

$$T_2 = \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} |G(x+\tau,s,\mu;\lambda+\delta) - G(x,s,\mu;\lambda+\delta)| ds d\lambda$$

$$= \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} \left[\frac{e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|z-s|}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} + \frac{e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|z+\tau-s|}}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \right] ds d\lambda$$

$$\leq \frac{M}{\sqrt{\lambda}} \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} \left[e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}|z-s|} + e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}|z+\tau-s|} \right] ds d\lambda$$

$$= \frac{M}{\sqrt{\lambda}} \tau^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_x^{x+\tau} \left[e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(s-x)} + e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(z+\tau-s)} \right] ds d\lambda.$$

$$= T_{21} + T_{22},$$

where

$$T_{21} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int\limits_0^{\infty}\lambda^{-\alpha}\int\limits_x^{x+\tau}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(s-x)}dsd\lambda,$$

$$T_{22} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int\limits_0^{\infty}\lambda^{-\alpha}\int\limits_x^{x+\tau}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(z+\tau-s)}dsd\lambda.$$

By changing variable,

$$\sqrt{\lambda}\frac{\sqrt{2}}{2}(s-x) = p,$$

we have

$$\lambda = \frac{2p^2}{(s-x)^2}, d\lambda = \frac{4pdp}{(s-x)^2}, \lambda^{-\left(\alpha+\frac{1}{2}\right)} = M\frac{(s-x)^{2\alpha+1}}{p^{2\alpha+1}}.$$

Then,

$$T_{21} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int\limits_x^{x+\tau}\int\limits_0^{\infty}\lambda^{-\alpha}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(s-x)}d\lambda ds \leq M_1\tau^{-2\alpha}\int\limits_x^{x+\tau}\int\limits_0^{\infty}\frac{(s-x)^{2\alpha+1}}{p^{2\alpha+1}}\frac{e^{-p}pdp}{(s-x)^2}ds$$

$$= M_1\tau^{-2\alpha}\int\limits_x^{x+\tau}(s-x)^{2\alpha-1}\,ds\int\limits_0^{\infty}e^{-p}p^{\,-2\alpha}\,dp$$

$$\leq M_2\frac{\tau^{-2\alpha}\tau^{2\alpha}}{2\alpha} = \frac{M_2}{2\alpha}.$$

Similarly, by changing variable,

$$\sqrt{\lambda}\frac{\sqrt{2}}{2}(x+\tau-s) = p,$$

we get

$$T_{21} = \frac{M}{\sqrt{\lambda}}\tau^{-2\alpha}\int\limits_x^{x+\tau}\int\limits_0^{\infty}\lambda^{-\alpha}e^{-\sqrt{\lambda}\frac{\sqrt{2}}{2}(x+\tau-s)}d\lambda ds$$

$$\leq M_1\tau^{-2\alpha}\int\limits_x^{x+\tau}\int\limits_0^{\infty}\frac{(x+\tau-s)^{2\alpha+1}}{p^{2\alpha+1}}\frac{e^{-p}pdp}{(x+\tau-s)^2}ds$$

$$\leq M_1\tau^{-2\alpha}\int\limits_x^{x+\tau}(x+\tau-s)^{2\alpha-1}\,ds\int\limits_0^{\infty}e^{-p}p^{\,-2\alpha}\,dp$$

$$= M_2\frac{\tau^{-2\alpha}\tau^{2\alpha}}{2\alpha} = \frac{M_2}{2\alpha}.$$

Thus,

$$T_2 \leq \frac{M_2}{2\alpha}.$$

Finally, let us estimate T_3.

$$T_3 = |\tau|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \int\limits_{x+\tau}^1 \int\limits_x^{x+\tau} |G_z(z,,s,\mu;\lambda+\delta)|\, dzdsd\lambda$$

$$\leq |\tau|^{-2\alpha} \int\limits_0^\infty \lambda^{-\alpha} \int\limits_{x+\tau}^1 \int\limits_x^{x+\tau} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|z-s|} dzdsd\lambda$$

$$\leq |\tau|^{-2\alpha} \int\limits_{x+\tau}^1 \int\limits_x^{x+\tau} \int\limits_0^\infty \lambda^{-\alpha} e^{-|\lambda|^{1/2}\frac{\sqrt{2}}{2}|z-s|} d\lambda dzds.$$

Using (2.46), we have

$$T_3 \leq M_1 \tau^{-2\alpha} \int\limits_{x+\tau}^1 \int\limits_x^{x+\tau} (s-z)^{2\alpha-2} dzds$$

$$\leq M_1 \tau^{-2\alpha} \int\limits_x^{x+\tau} \frac{(x+\tau-z)^{2\alpha-1}}{1-2\alpha} dz$$

$$\leq M_1 \frac{\tau^{-2\alpha}\tau^{2\alpha}}{(1-2\alpha)\,2\alpha} = \frac{M_1}{(1-2\alpha)\,2\alpha}. \tag{2.48}$$

Finally,

$$T \leq \frac{M}{1-2\alpha} + \frac{M}{2\alpha} \leq \frac{M}{2\alpha\,(1-2\alpha)}.$$

Theorem 2.2.1 is proved.

From positivity of A^x in $E_{\alpha,\infty}$ and Theorem 2.2.1 it follows the positivity of A^x on $\overset{\circ}{C}{}^{2\alpha}[0,1]$.

Theorem 2.2.2. *The operator $(\lambda + A^x)$ has a bounded inverse in $\overset{\circ}{C}{}^{2\alpha}[0,1]$ for any $\lambda \geq 0$ and the following estimate holds:*

$$\left\|(\lambda + A^x)^{-1}\right\|_{\overset{\circ}{C}{}^{2\alpha}[0,1]\to \overset{\circ}{C}{}^{2\alpha}[0,1]} \leq \frac{M(\delta)}{2\alpha(1-2\alpha)} \frac{M}{\delta+\lambda}.$$

2.3 APPLICATIONS

In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for parabolic and elliptic equations.

2.3.1 Parabolic problems

First, we consider the initial boundary value problem

$$\begin{cases} -\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \delta u(t,x) = f(t,x), 0 < t < T, x \in (0,1)\,, \\ u(0,x) = \varphi(x), x \in [0,1], \\ u(t,0) = 0, u(t,1) = u(t,\mu), 0 \le \mu < 1, 0 \le t \le T. \end{cases} \tag{2.49}$$

Here, $\varphi(x)$ and $f(t,x)$ are sufficiently smooth functions and they satisfy compatibility conditions which guarantee problem (2.49) has a smooth solution $u(t,x)$.

Theorem 2.3.1. *Let $0 < 2\alpha < 1$. Then, for the solution of the initial value problem (2.49), we have the following coercive stability inequality*

$$\|u_t\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,1]\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha+2}[0,1]\right)}$$

$$\le M(\alpha)\left[\|\varphi\|_{\overset{\circ}{C}^{2\alpha+2}[0,1]} + \|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,1]\right)}\right].$$

The proof of Theorem 2.3.1 is based on Theorem 2.2.1 on the structure of the fractional spaces $E_\alpha = E_{\alpha,\infty}\left(C\left[0,1\right], A^x\right)$, on the Theorem 2.1.1 on the positivity of the operator A^x, on the following theorem on coercive stability of initial value for the abstract parabolic equation.

Theorem 2.3.2. *(Ashyralyev and Sobolevskii, 2004) Let A be a strongly positive operator in a Banach space E and $\varphi \in D\left(A\right), f \in C\left(\left[0,T\right], E_\alpha\right), 0 < \alpha < 1$. Then, the solution of the initial value problem*

$$u' + Au(t) = f(t), 0 < t < T, u(0) = \varphi \tag{2.50}$$

in a Banach space E, satisfies the following coercive inequality

$$\|u'\|_{C([0,T],E_\alpha)} + \|Au\|_{C([0,T],E_\alpha)}$$

$$\le M\left[\|A\varphi\|_{E_\alpha} + \frac{M}{\alpha(1-\alpha)}\|f\|_{C([0,T],E_\alpha)}\right].$$

Second, we consider the nonlocal boundary value problem for the parabolic equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x), 0 < t < T, x \in (0,1)\,, \\ u(0,x) = u(T,x), x \in [0,1], \\ u(t,0) = 0, u(t,1) = u(t,\mu), 0 \leq \mu < 1, 0 \leq t \leq T. \end{cases} \tag{2.51}$$

Here, $f(t,x)$ is a sufficiently smooth function and it satisfies any compatibility conditions which guarantee problem (2.51) has a smooth solution $u(t,x)$.

Theorem 2.3.3. *Let $0 < 2\alpha < 1$. Then, for the solution of boundary value problem (2.51), the following coercive stability inequality holds:*

$$\|u_t\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,1]\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha+2}[0,1]\right)} \leq M(\alpha)\,\|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,1]\right)}\,.$$

The proof of Theorem 2.3.3 is based on Theorem 2.2.1 on the structure of the fractional spaces $E_\alpha = E_{\alpha,\infty}\left(C\,[0,1]\,, A^x\right),$

Theorem 2.1.1 on the positivity of the operator A^x on the following theorem on the coercive stability of the nonlocal boundary value for the abstract parabolic equation.

Theorem 2.3.4. *(Ashyralyev and Sobolevskii, 2004) Let A be a strongly positive operator in a Banach space E and $f \in C\left([0,T]\,, E_\alpha\right), 0 < \alpha < 1$. Then, for the solution of the nonlocal boundary value problem*

$$u' + Au(t) = f(t), 0 < t < T, u(0) = u(T) \tag{2.52}$$

in a Banach space E, we have the following coercive inequality

$$\|u'\|_{C([0,T],E_\alpha)} + \|Au\|_{C([0,T],E_\alpha)} \leq \frac{M}{\alpha(1-\alpha)}\,\|f\|_{C([0,T],E_\alpha)}\,.$$

2.3.2 Elliptic problems

First, we consider the boundary value problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x), \quad 0 < t < T, \ x \in (0,1), \\ u(0,x) = \varphi(x), \ u(T,x) = \psi(x), \quad x \in [0,1], \\ u(t,0) = 0, u(t,1) = u(t,\mu), 0 \leq \mu < 1, 0 \leq t \leq T. \end{cases} \tag{2.53}$$

Here, $\varphi(x), \psi(x)$ and $f(t,x)$ are sufficiently smooth functions and they satisfy any compatibility conditions which guarantee problem (2.53) has a smooth solution $u(t,x)$.

Theorem 2.3.5. *Let $0 < 2\alpha < 1$. Then, for the solution of the boundary value problem (2.53), the following coercive stability inequality*

$$\|u_{tt}\|_{C\left([0,T],\overset{\circ}{C}{}^{2\alpha}[0,1]\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}{}^{2+2\alpha}[0,1]\right)}$$

$$\leq M(\alpha)\left[\|\varphi\|_{\overset{\circ}{C}{}^{2+2\alpha}[0,1]} + \|\psi\|_{\overset{\circ}{C}{}^{2+2\alpha}[0,1]} + \|f\|_{C\left([0,T],\overset{\circ}{C}{}^{2+2\alpha}[0,1]\right)}\right]$$

is valid.

The proof of Theorem 2.3.5 is based on Theorem 2.2.1 on the structure of the fractional spaces $E_\alpha = E_{\alpha,\infty}(C[0,1], A^x)$, Theorem 2.1.1 on the positivity of the operator A^x, on the following theorems on coercive stability of boundary value problem for the abstract elliptic equation and on the structure of the fractional space $E'_\alpha = E_\alpha(E, A^{1/2})$ which is the Banach space consisting of those $v \in E$ for which the norm

$$\|v\|_{E'_\alpha} = \sup_{\lambda > 0} \lambda^\alpha \left\| A^{1/2}\left(\lambda + A^{1/2}\right)^{-1} v \right\|_E + \|v\|_E$$

is finite.

Theorem 2.3.6. *(Ashyralyev and Sobolevskii, 2004) The spaces $E_\alpha(E,A)$ and $E'_{2\alpha}(E,A^{1/2})$ coincide for any $0 < \alpha < \frac{1}{2}$, and their norms are equivalent.*

Theorem 2.3.7. *(Ashyralyev and Sobolevskii, 2004) Let A be positive operator in a Banach space E and* $\varphi \in D(A), \psi \in D(A),\ f \in C([0,T], E'_\alpha), 0 < \alpha < 1$. *Then, for the solution of the boundary value problem*

$$-u'' + Au(t) = f(t),\ 0 < t < T, u(0) = \varphi, \quad u(T) = \psi \tag{2.54}$$

in a Banach space E the following coercive inequality holds:

$$\|u''\|_{C([0,T],E'_\alpha)} + \|Au\|_{C([0,T],E'_\alpha)} \le M\left[\|A\varphi\|_{E'_\alpha} + \|A\psi\|_{E'_\alpha} + \frac{M}{\alpha(1-\alpha)}\|f\|_{C([0,T],E'_\alpha)}\right].$$

Second, we consider the nonlocal boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial^2 u(t,x)}{\partial x^2} + \delta u(t,x) = f(t,x), \quad 0 < t < T, \quad x \in (0,1), \\ u(0,x) = u(T,x), \quad u_t(0,x) = u_t(T,x), \quad x \in [0,1], \\ u(t,0) = 0, u(t,1) = u(t,\mu), 0 \le \mu < 1, 0 \le t \le T. \end{cases} \tag{2.55}$$

Here, $f(t,x)$ is sufficiently smooth function and it satisfies any compatibility conditions which guarantee problem (2.55) has a smooth solution $u(t,x)$.

Theorem 2.3.8. *Let* $0 < 2\alpha < 1$. *Then, for the solution of nonlocal boundary value problem (2.55), the following coercive stability inequality holds:*

$$\|u_{tt}\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,1]\right)} + \|u\|_{C\left([0,T],\overset{\circ}{C}^{2+2\alpha}[0,1]\right)} \le M(\alpha)\|f\|_{C\left([0,T],\overset{\circ}{C}^{2\alpha}[0,1]\right)}.$$

The proof of Theorem 2.3.8 is based on Theorem 2.2.1 on the structure of the fractional spaces $E_\alpha = E_{\alpha,\infty}(C[0,1], A^x)$, Theorem 2.1.1 on the positivity of the operator A^x, Theorem 2.3.6 on the structure of the fractional space $E'_\alpha = E_\alpha(E, A^{1/2})$ and on the following theorem on coercive stability of the nonlocal boundary value problem for the abstract elliptic equation.

Theorem 2.3.9. *(Ashyralyev, 2003) Let A be a positive operator in a Banach space E and* $f \in C([0,T], E'_\alpha)$, $0 < \alpha < 1$. *Then, for the solution of the nonlocal boundary value problem*

$$\begin{cases} -u'' + Au(t) = f(t),\ 0 < t < T, \\ u(0) = u(T), u'(0) = u'(T) \end{cases} \tag{2.56}$$

in a Banach space E, the coercive inequality

$$\|u''\|_{C([0,T],E'_\alpha)} + \|Au\|_{C([0,T],E'_\alpha)} \leq \frac{M}{\alpha(1-\alpha)} \|f\|_{C([0,T],E'_\alpha)}$$

is valid.

CHAPTER 3

POSITIVITY OF A^X AND STRUCTURE OF FRACTIONAL SPACES $E_{\alpha,1}(L_1[0,1], A^X)$

In Chapter 3, the positivity of the differential operator A^x in $L_1\,[0,1]$ is established. The structure of fractional spaces $E_{\alpha,1}(L_1[0,1], A^x)$ will be investigated. It is established that for any $0 < \alpha < 1/2$ the norms in the spaces $E_{\alpha,1}(L_1[0,1], A^x)$ and $W_1^{2\alpha}\,[0,1]$ are equivalent. This result allows us to prove the positivity of A^x in $W_1^{2\alpha}\,[0,1]\,(0 < \alpha < 1/2)$. In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for elliptic and parabolic equations.

3.1 POSITIVITY OF A^X IN $L_1\,[0,1]$

First, the positivity of A^x in $L_1\,[0,1]$ is investigated.

Theorem 3.1.1. *For all $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| \leq \varphi, \varphi < \pi/2\}$, the resolvent $(\lambda I + A^x)^{-1}$ defined by formula (2.4) is subject to the bound*

$$\left\|(\lambda I + A^x)^{-1}\right\|_{L_1[0,1]\to L_1[0,1]} \leq \frac{M(\varphi,\delta)}{1+|\lambda|}.$$

Proof. Using formula (2.4) and the triangle inequality, we get

$$|u(x)| \leq \int_0^1 |G(x,s,\mu;\lambda+\delta)|\,|f(s)|\,ds \tag{3.1}$$

for any $x \in [0,1]$. We denote that

$$f_*(s) = \begin{cases} f(s), s \in [0,1], \\ \\ 0, s \notin [0,1]. \end{cases} \tag{3.2}$$

Then, using inequality (3.1), estimates (2.7), (2.8) and the triangle inequality, we have

$$\int_0^1 |u(x)|\, dx \le \int_0^1 \int_0^1 |G(x,s,\mu;\lambda+\delta)|\, |f_*(s)|\, dsdx$$

$$= \int_{\frac{1+\mu}{2}}^1 \int_0^{\mu} |G(x,s,\mu;\lambda+\delta)|\, |f_*(s)|\, dsdx + \int_{\frac{1+\mu}{2}}^1 \int_{\mu}^1 |G(x,s,\mu;\lambda+\delta)|\, |f_*(s)|\, dsdx$$

$$+ \int_0^{\frac{1+\mu}{2}} \int_0^1 |G(x,s,\mu;\lambda+\delta)|\, |f_*(s)|\, dsdx$$

$$\le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{\frac{1+\mu}{2}}^1 \int_0^{\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)} |f_*(s)|\, dsdx$$

$$+ \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{\frac{1+\mu}{2}}^1 \int_{\mu}^{x+\frac{\mu-1}{2}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)} |f_*(s)|\, dsdx$$

$$+ \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{\frac{1+\mu}{2}}^1 \int_{x+\frac{\mu-1}{2}}^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} |f_*(s)|\, dsdx$$

$$+ \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_0^{\frac{1+\mu}{2}} \int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} |f_*(s)|\, dsdx$$

$$\le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{\frac{1+\mu}{2}}^1 \int_0^{1+\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-y+\mu)} |f_*(y-x)|\, dydx$$

$$+ \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{\frac{1+\mu}{2}}^1 \int_{\mu-1}^{\frac{\mu-1}{2}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1+y-\mu)} |f_*(y+x)|\, dydx$$

$$+ \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \int_{\frac{1+\mu}{2}}^1 \int_{\frac{\mu-1}{2}}^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|} |f_*(y+x)|\, dydx$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_0^{\frac{1+\mu}{2}}\int\limits_{-1}^{1}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}\left|f_*(y+x)\right|dydx$$

$$\leq\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_0^{1+\mu}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-y+\mu)}dy\int\limits_{\frac{1+\mu}{2}}^{1}\left|f_*(y-x)\right|dx$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\mu-1}^{\frac{\mu-1}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1+y-\mu)}dy\int\limits_{\frac{1+\mu}{2}}^{1}\left|f_*(y+x)\right|dx$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\frac{1+\mu}{2}}^{1}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\int\limits_{\frac{\mu-1}{2}}^{1}\left|f_*(y+x)\right|dx$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_0^{\frac{1+\mu}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\int\limits_{-1}^{1}\left|f_*(y+x)\right|dx$$

$$\leq\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_0^{1+\mu}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-y+\mu)}dy\int\limits_0^{1}\left|f_*(s)\right|ds$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\mu-1}^{\frac{\mu-1}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1+y-\mu)}dy\int\limits_0^{1}\left|f_*(s)\right|ds$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\frac{1+\mu}{2}}^{1}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\int\limits_0^{1}\left|f_*(s)\right|ds$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_0^{\frac{1+\mu}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\int\limits_{-1}^{1}\left|f_*(s)\right|ds$$

$$\leq\frac{M(\varphi,\delta)}{1+|\lambda|}\left\|f\right\|_{L_1[0,1]}.$$

Thus, we obtain

$$\left\|(A^x+\lambda)^{-1}f\right\|_{L_1[0,1]}\leq\frac{M(\varphi,\delta)}{1+|\lambda|}\left\|f\right\|_{L_1[0,1]}.$$

From that it follows

$$\left\|(A^x+\lambda)^{-1}\right\|_{L_1[0,1]\to L_1[0,1]}\leq\frac{M(\varphi,\delta)}{1+|\lambda|}.$$

So, Theorem 3.1.1 is proved.

Clearly, the operators A^x and its resolvent $(\lambda + A^x)^{-1}$ commute. Thus, from the definition of the norm in the space $E_{\alpha,1}(L_1[0,1], A^x)$ it follows that

$$\left\|(\lambda + A^x)^{-1}\right\|_{E_{\alpha,1}(L_1[0,1],A^x)\to E_{\alpha,1}(L_1[0,1],A^x)} \le \left\|(\lambda + A^x)^{-1}\right\|_{L_1[0,1]\to L_1[0,1]}.$$

Hence, by using Theorem 3.1.1, we obtain the positivity of the operator A^x in the fractional spaces $E_{\alpha,1}(L_1[0,1], A^x)$.

3.2 THE STRUCTURE OF FRACTIONAL SPACES $E_{\alpha,1}(L_1[0,1], A^X)$, POSITIVITY OF A^X -IN- $\overset{\circ}{W}_1^{2\alpha}[0,1]$

Now, we will study the positivity of A^x in $\overset{\circ}{W}_1^{2\alpha}[0,1]$. We have the following theorem.

Theorem 3.2.1. *Let $\alpha \in (0, 1/2)$. Then, the norms of the spaces $E_{\alpha,1}(L_1[0,1], A^x)$ and $\overset{\circ}{W}_1^{2\alpha}[0,1]$ are equivalent.*

Here, $\overset{\circ}{W}_1^{2\alpha}[0,1]$ $(0 < 2\alpha < 1)$ is the Banach space of all integrable functions $\varphi(x)$ defined on $[0,1]$ and satisfying a Hölder condition for which the following norm is finite

$$\|\varphi\|_{\overset{\circ}{W}_1^{2\alpha}[0,1]} = \int_0^1\int_0^1 \frac{|\varphi(x+y) - \varphi(x)|}{y^{1+2\alpha}}dydx + \int_0^1 \frac{|\varphi(x) - \varphi(0)|}{x^{2\alpha}}dx + \|\varphi\|_{L_1[0,1]}.$$

Proof. For any $\lambda > 0$ using formula (2.38), the triangle inequality, we get

$$\int_0^\infty\int_0^1 \lambda^\alpha \left|A^x(\lambda + A^x)^{-1}f(x)\right| dx\frac{d\lambda}{\lambda} \le \int_0^\infty \frac{\delta\lambda^\alpha}{\delta+\lambda}\frac{d\lambda}{\lambda}\int_0^1 |f(x)|\, dx$$
$$+\int_0^\infty \frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1 \Big[e^{-\sqrt{\delta+\lambda}x} + e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})$$
$$\times (1 + e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big]\, |f(x)|\, dx\frac{d\lambda}{\lambda}$$
$$+\int_0^1\int_0^\infty \lambda^{\alpha+1}\int_0^1 |G(x,s,\mu;\lambda+\delta)|\, |f(x) - f(s|\, dsdx\frac{d\lambda}{\lambda}$$
$$= P_1 + P_2 + P_3,$$

where

$$P_1 = \int_0^\infty \frac{\delta\lambda^\alpha}{\delta+\lambda}\frac{d\lambda}{\lambda}\int_0^1 |f(x)|\,dx,$$

$$P_2 = \int_0^\infty \frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1 \Big[e^{-\sqrt{\delta+\lambda}x} + e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})$$
$$\times (1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big]\,|f(x)|\,dx\frac{d\lambda}{\lambda},$$
$$P_3 = \int_0^\infty \lambda^{\alpha+1}\int_0^1\int_0^1 |G(x,s,\mu;\lambda+\delta)|\,|f(x)-f(s|\,dsdx\frac{d\lambda}{\lambda}.$$

Using the definition of norm space $\overset{\circ}{W}{}_1^{2\alpha}[0,1]$, we have

$$P_1 \leq \int_0^\infty \frac{\delta\lambda^\alpha}{\delta+\lambda}\frac{d\lambda}{\lambda}\,\|f\|_{L_1[0,1]} \leq \left[\int_0^1 \frac{d\lambda}{\lambda^{1-\alpha}} + \delta\int_1^\infty \frac{d\lambda}{\lambda^{2-\alpha}}\right]\|f\|_{\overset{\circ}{W}{}_1^{2\alpha}[0,1]}$$

$$\leq \frac{M(\delta)}{\alpha(1-\alpha)}\,\|f\|_{\overset{\circ}{W}{}_1^{2\alpha}[0,1]}. \tag{3.3}$$

Now, we will estimate P_2. We obtain that

$$P_2 \leq \int_0^\infty \frac{\lambda^\alpha}{\delta+\lambda}\int_0^1 e^{-\sqrt{\delta+\lambda}x}\,|f(x)-f(0)|\,dxd\lambda$$

$$+\int_0^\infty \frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1 \Big[e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})$$
$$\times (1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big]\,|f(x)|\,dx\frac{d\lambda}{\lambda} = P_{21}+P_{22},$$

where

$$P_{21} = \int_0^\infty \frac{\lambda^\alpha}{\delta+\lambda}\int_0^1 e^{-\sqrt{\delta+\lambda}x}\,|f(x)-f(0)|\,dxd\lambda,$$

$$P_{22} = \int_0^\infty \frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1 \Big[e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)} - e^{-\sqrt{\delta+\lambda}(1+x)})(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big]\,|f(x)|\,dx\frac{d\lambda}{\lambda}.$$

First, we will estimate P_{21}. We have that

$$P_{21} = \int_0^1 |f(x)-f(0)|\int_0^\infty \frac{\lambda^\alpha}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x}d\lambda dx.$$

Since

$$\int_0^\infty \frac{\lambda^\alpha}{\delta+\lambda} e^{-\sqrt{\delta+\lambda}x} d\lambda \leq \int_0^\infty (\delta+\lambda)^{\alpha-1} e^{-\sqrt{\delta+\lambda}x} d\lambda$$

$$= \int_{\sqrt{\delta}x}^\infty y^{\alpha-1} e^{-y} dy x^{2-2\alpha-2} \leq \int_0^\infty y^{\alpha-1} e^{-y} dy x^{-2\alpha} = G(\alpha) x^{-2\alpha},$$

we get

$$P_{21} \leq G(\alpha) \int_0^1 x^{-2\alpha} \left| f(x) - f(0) \right| dx \leq G(\alpha) \left\| f \right\|_{\overset{\circ}{W_1}{}^{2\alpha}[0,1]}. \tag{3.4}$$

Second, we estimate P_{22}. Clearly, we have

$$P_{22} = M(\delta) \int_0^1 |f(x)| \int_0^\infty (\lambda+\delta)^{\alpha-1} e^{-\sqrt{\delta+\lambda}(1+\mu-x)} d\lambda dx.$$

In same manner, we obtain

$$\int_0^\infty \frac{\lambda^\alpha}{\delta+\lambda} e^{-\sqrt{\delta+\lambda}(1+\mu-x)} d\lambda \leq \int_0^\infty (\delta+\lambda)^{\alpha-1} e^{-\sqrt{\delta+\lambda}(1+\mu-x)} d\lambda$$

$$= \int_{\sqrt{\delta}(1+\mu-x)}^\infty y^{\alpha-1} e^{-y} dy (1+\mu-x)^{2-2\alpha-2}$$

$$\leq \int_0^\infty y^{\alpha-1} e^{-y} dy (1+\mu-x)^{-2\alpha} = G(\alpha)(1+\mu-x)^{-2\alpha}.$$

Thus, we get

$$P_{22} \leq M(\delta) G(\alpha) \int_0^1 (1+\mu-x)^{-2\alpha} |f(x)| dx$$

$$\leq M(\delta) G(\alpha) \int_0^1 \mu^{-2\alpha} |f(x)| dx \leq M(\delta,\mu) G(\alpha) \left\| f \right\|_{\overset{\circ}{W_1}{}^{2\alpha}[0,1]}. \tag{3.5}$$

Combining estimates (3.4) and (3.5), we have

$$P_2 \leq \left(M(\delta,\mu) + 1 \right) G(\alpha) \left\| f \right\|_{\overset{\circ}{W_1}{}^{2\alpha}[0,1]}. \tag{3.6}$$

Now, we will estimate P_3.

$$P_3 = \int_0^\infty \lambda^{\alpha+1} \int_0^1 \int_0^1 \left| G(x,s,\mu;\lambda+\delta) \right| \left| f(x) - f(s) \right| ds dx \frac{d\lambda}{\lambda}$$

$$\leq \int_0^{\frac{1+\mu}{2}} \int_0^1 \int_0^\infty \lambda^{\alpha+1} \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \frac{d\lambda}{\lambda} |f(x)-f(s)|\, dsdx$$

$$+ \int_{\frac{1+\mu}{2}}^{1} \int_0^\mu \int_0^\infty \lambda^{\alpha+1} \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)} \frac{d\lambda}{\lambda} |f(x)-f(s)|\, dsdx$$

$$+ \int_{\frac{1+\mu}{2}}^{1} \int_\mu^{x+\frac{\mu-1}{2}} \int_0^\infty \lambda^{\alpha+1} \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)} \frac{d\lambda}{\lambda} |f(x)-f(s)|\, dsdx$$

$$+ \int_{\frac{1+\mu}{2}}^{1} \int_{x+\frac{\mu-1}{2}}^{1} \int_0^\infty \lambda^{\alpha+1} \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \frac{d\lambda}{\lambda} |f(x)-f(s)|\, dsdx.$$

We have that

$$\int_0^\infty \lambda^{\alpha+1} \frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \frac{d\lambda}{\lambda}$$

$$\leq \int_0^\infty \frac{1}{\lambda^{\frac{1}{2}-\alpha}} e^{-\lambda^{\frac{1}{2}}\frac{\sqrt{2}}{2}|x-s|} d\lambda.$$

Putting $\lambda^{\frac{1}{2}}\frac{\sqrt{2}}{2}|x-s| = y$, we get $\lambda = \frac{2y^2}{|x-s|^2}$ and $d\lambda = \frac{4ydy}{|x-s|^2}$. Then,

$$\int_0^\infty \frac{1}{\lambda^{\frac{1}{2}-\alpha}} e^{-\lambda^{\frac{1}{2}}\frac{\sqrt{2}}{2}|x-s|} d\lambda = \int_0^\infty \left(\frac{2y^2}{|x-s|^2}\right)^{\alpha-\frac{1}{2}} e^{-y} \frac{4ydy}{|x-s|^2}$$

$$= 2^{\alpha+\frac{3}{2}} \frac{1}{|x-s|^{2\alpha+1}} \int_0^\infty e^{-y} y^{2\alpha} dy = G(2\alpha+1) \frac{2^{\alpha+\frac{3}{2}}}{|x-s|^{2\alpha+1}}$$

and

$$\int_0^\infty \lambda^{\alpha+1} \frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|} \frac{d\lambda}{\lambda} \leq G(2\alpha+1) \frac{2^{\alpha+\frac{3}{2}}}{|x-s|^{2\alpha+1}}.$$

Similarly, we can show that

$$\int_0^\infty \lambda^{\alpha+1} \frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)} \frac{d\lambda}{\lambda} \leq G(2\alpha+1) \frac{2^{\alpha+\frac{3}{2}}}{(1-x+s-\mu)^{2\alpha+1}}$$

and

$$\int_0^\infty \lambda^{\alpha+1} \frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)} \frac{d\lambda}{\lambda} \leq G(2\alpha+1) \frac{2^{\alpha+\frac{3}{2}}}{(1-x+\mu-s)^{2\alpha+1}}.$$

Therefore,

$$P_3 \leq G(2\alpha+1)M_2(\delta,\mu)\int\limits_0^{\frac{1+\mu}{2}}\int\limits_0^1 \frac{|f(x)-f(s)|}{|x-s|^{2\alpha+1}}dsdx$$

$$+G(2\alpha+1)M_2(\delta,\mu)\int\limits_{\frac{1+\mu}{2}}^1\int\limits_0^\mu \frac{|f(x)-f(s)|}{(1-x+\mu-s)^{2\alpha+1}}dsdx$$

$$+G(2\alpha+1)M_2(\delta,\mu)\int\limits_{\frac{1+\mu}{2}}^1\int\limits_\mu^{x+\frac{\mu-1}{2}} \frac{|f(x)-f(s)|}{(1-x+s-\mu)^{2\alpha+1}}dsdx$$

$$+G(2\alpha+1)M_2(\delta,\mu)\int\limits_{\frac{1+\mu}{2}}^1\int\limits_{x+\frac{\mu-1}{2}}^1 \frac{|f(x)-f(s|}{|x-s|^{2\alpha+1}}dsdx$$

$$= G(2\alpha+1)M_2(\delta,\mu)\left\{\int\limits_0^{\frac{1+\mu}{2}}\int\limits_0^1 \frac{|f(x)-f(s)|}{|x-s|^{2\alpha+1}}dsdx\right.$$

$$+\int\limits_{\frac{1+\mu}{2}}^1\int\limits_0^\mu \frac{|f(x)-f(s)|}{(1-x+\mu-s)^{2\alpha+1}}dsdx+\int\limits_{\frac{1+\mu}{2}}^1\int\limits_\mu^{x+\frac{\mu-1}{2}} \frac{|f(x)-f(s)|}{(1-x+s-\mu)^{2\alpha+1}}dsdx$$

$$\left.+\int\limits_{\frac{1+\mu}{2}}^1\int\limits_{x+\frac{\mu-1}{2}}^1 \frac{|f(x)-f(s|}{|x-s|^{2\alpha+1}}dsdx\right\}$$

$$\leq G(2\alpha+1)M_3(\delta,\mu)\,\|f\|_{\overset{\circ}{W}_1^{2\alpha}[0,1]}.$$

Thus,

$$\|f\|_{E_{\alpha,1}(L_1[0,1],A^x)} \leq M_4(a,\mu,\delta)\,\|f\|_{\overset{\circ}{W}_1^{2\alpha}[0,1]}. \tag{3.7}$$

Now, let us prove the reverse inequality

$$\|f\|_{\overset{\circ}{W}_1^{2\alpha}[0,1]} \leq M_4(a,\mu,\delta)\,\|f\|_{E_{\alpha,1}(L_1[0,1],A^x)}. \tag{3.8}$$

Applying formula (2.44), we get

$$f(x+\tau)-f(x)$$

$$=\int\limits_0^\infty \lambda^{-\alpha}\int\limits_0^1 \left[G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right]\lambda^\alpha A^s(\lambda+A^s)^{-1}f(s)dsd\lambda.$$

The proof estimate (3.8) is based on this formula and estimates (2.7), (2.8) and the triangle inequality. This finishes the proof of Theorem 3.2.1.

From positivity of A^x in $E_{\alpha,1}$ and Theorem 3.2.1 it follows the positivity of A^x on $\overset{\circ}{W}_1^{2\alpha}[0,1]$.

Theorem 3.2.2. *The operator* $(\lambda + A^x)$ *has a bounded inverse in* $\overset{\circ}{W}_1^{2\alpha}[0,1]$ *for any* $\lambda \geq 0$ *and the following estimate holds:*

$$\left\|(\lambda + A^x)^{-1}\right\|_{\overset{\circ}{W}_1^{2\alpha}[0,1] \to \overset{\circ}{W}_1^{2\alpha}[0,1]} \leq \frac{M(\delta)}{2\alpha(1-2\alpha)} \frac{M}{\delta + \lambda}.$$

3.3 APPLICATIONS

In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for parabolic and elliptic equations.

3.3.1 Parabolic problems

First, we consider the initial boundary value problem (2.49).

Theorem 3.3.1. *Let* $0 < 2\alpha < 1$. *Then, for the solution of the initial value problem (2.49), we have the following coercive stability inequality*

$$\|u_t\|_{L_1\left([0,T], \overset{\circ}{W}_1^{2\alpha}[0,1]\right)} + \|u\|_{L_1\left([0,T], \overset{\circ}{W}_1^{2+2\alpha}[0,1]\right)}$$

$$\leq M(\alpha) \left[\|\varphi\|_{\overset{\circ}{W}_1^{2+2\alpha}[0,1]} + \|f\|_{L_1\left([0,T], \overset{\circ}{W}_1^{2\alpha}[0,1]\right)} \right].$$

The proof of Theorem 3.3.1 is based on Theorem 3.2.1 on the structure of the fractional spaces $E_{\alpha,1} = E_{\alpha,1}(L_1[0,1], A^x)$, Theorem 3.1.1 on the positivity of the operator A^x on the following theorem on coercive stability of initial value for the abstract parabolic equation.

Theorem 3.3.2. *Let* A *be a strongly positive operator in a Banach space* E *and* $\varphi \in D(A)$, $f \in L_1([0,T], E_{\alpha,1})$, $0 < \alpha < 1$. *Then, for the solution of the initial value problem (2.50), the following coercive inequality*

$$\|u'\|_{L_1([0,T],E_{\alpha,1})} + \|Au\|_{L_1([0,T],E_{\alpha,1})}$$

$$\leq M \left[\|A\varphi\|_{E_{\alpha,1}} + \frac{M}{\alpha(1-\alpha)} \|f\|_{L_1([0,T],E_{\alpha,1})} \right]$$

is valid.

Second, we consider the nonlocal boundary value problem for parabolic equation (2.51).

Theorem 3.3.3. *Let* $0 < 2\alpha < 1$. *Then, for the solution of boundary value problem (2.51), the following coercive stability inequality holds:*

$$\|u_t\|_{L_1\left([0,T],\overset{\circ}{W}_1^{2\alpha}[0,1]\right)} + \|u\|_{L_1\left([0,T],\overset{\circ}{W}_1^{2+2\alpha}[0,1]\right)} \leq M(\alpha) \|f\|_{L_1\left([0,T],\overset{\circ}{W}_1^{2\alpha}[0,1]\right)}.$$

The proof of Theorem 3.3.3 is based on Theorem 3.2.1 on the structure of the fractional spaces $E_{\alpha,1} = E_{\alpha,1}(L_1[0,1], A^x)$,

Theorem 3.1.1 on the positivity of the operator A^x on the following theorem on the coercive stability of the nonlocal boundary value for the abstract parabolic equation.

Theorem 3.3.4. *Let A be a strongly positive operator in a Banach space E and $f \in L_1([0,T], E_{\alpha,1}), 0 < \alpha < 1$. Then, for the solution of the nonlocal boundary value problem(2.52), we have the following coercive inequality*

$$\|u'\|_{L_1([0,T],E_{\alpha,1})} + \|Au\|_{L_1([0,T],E_{\alpha,1})} \leq \frac{M}{\alpha(1-\alpha)} \|f\|_{L_1([0,T],E_{\alpha,1})}.$$

3.3.2 Elliptic problems

First, we consider the boundary value problem (2.53).

Theorem 3.3.5. *Let* $0 < 2\alpha < 1$. *Then, the solution of the boundary value problem (2.53) satisfies the following coercive stability inequality*

$$\|u_{tt}\|_{L_1\left([0,T],\overset{\circ}{W}_1^{2\alpha}[0,1]\right)} + \|u\|_{L_1\left([0,T],\overset{\circ}{W}_1^{2+2\alpha}[0,1]\right)}$$

$$\leq M(\alpha) \left[\|\varphi\|_{\overset{\circ}{W}_1^{2+2\alpha}[0,1]} + \|\psi\|_{\overset{\circ}{W}_1^{2+2\alpha}[0,1]} + \|f\|_{L_1\left([0,T],\overset{\circ}{W}_1^{2\alpha}[0,1]\right)} \right].$$

The proof of Theorem 3.3.5 is based on Theorem 3.2.1 on the structure of fractional spaces $E_{\alpha,1} = E_{\alpha,1}(L_1[0,1], A^x)$, Theorem 3.1.1 on the positivity of the operator A^x, on the following theorems on coercive stability of boundary value problem for the abstract elliptic equation and on the structure of the fractional space $E'_{\alpha,1} = E_{\alpha,1}(E, A^{1/2})$ which is the Banach space consisting of those $v \in E$ for which the norm

$$\|v\|_{E'_{\alpha,1}} = \int_0^\infty \lambda^\alpha \left\| A^{1/2} \left(\lambda + A^{1/2}\right)^{-1} v \right\|_E \frac{d\lambda}{\lambda}$$

is finite.

Theorem 3.3.6. *(Ashyralyev and Sobolevskii, 2004) The spaces $E_{\alpha,1}(E, A)$ and $E'_{2\alpha,1}(E, A^{1/2})$ coincide for any $0 < \alpha < \frac{1}{2}$, and their norms are equivalent.*

Theorem 3.3.7. *(Ashyralyev and Sobolevskii, 2004) Let A be positive operator in a Banach space E and $\varphi \in D(A), \psi \in D(A)$, $f \in L_1([0,T], E'_{\alpha,1}), 0 < \alpha < 1$. Then, for the solution of the boundary value problem (2.54), the following coercive inequality holds*

$$\|u''\|_{L_1([0,T],E'_{\alpha,1})} + \|Au\|_{L_1([0,T],E'_{\alpha,1})} \le M \left[\|A\varphi\|_{E'_{\alpha,1}} + \|A\psi\|_{E'_{\alpha,1}} + \frac{M}{\alpha(1-\alpha)} \|f\|_{L_1([0,T],E'_{\alpha,1})} \right].$$

Second, we consider the nonlocal boundary value problem for the elliptic equation (2.55).

Theorem 3.3.8. *Let $0 < 2\alpha < 1$. Then, for the solution of nonlocal boundary value problem (2.55), we have the following coercive stability inequality*

$$\|u_{tt}\|_{L_1\left([0,T], \overset{\circ}{W}{}_1^{2\alpha}[0,1]\right)} + \|u\|_{L_1\left([0,T], \overset{\circ}{W}{}_1^{2+2\alpha}[0,1]\right)} \le M(\alpha) \|f\|_{L_1\left([0,T], \overset{\circ}{W}{}_1^{2\alpha}[0,1]\right)}.$$

The proof of Theorem 3.3.8 is based on Theorem 3.2.1 on the structure of the fractional spaces $E_{\alpha,1} = E_{\alpha,1}(L_1[0,1], A^x)$, Theorem 3.1.1 on the positivity of the operator A^x, Theorem 3.3.6 on the structure of the fractional space $E'_{\alpha,1} = E_{\alpha,1}(E, A^{1/2})$ and on the following theorem on coercive stability of the nonlocal boundary value problem for the abstract elliptic equation.

Theorem 3.3.9. *(Ashyralyev, 2003) Let A be a positive operator in a Banach space E and $f \in L_1([0,T], E'_{\alpha,1})$, $0 < \alpha < 1$. Then, for the solution of the nonlocal boundary value problem (2.56), the coercive inequality*

$$\|u''\|_{L_1([0,T],E'_{\alpha,1})} + \|Au\|_{L_1([0,T],E'_{\alpha,1})} \le \frac{M}{\alpha(1-\alpha)} \|f\|_{L_1([0,T],E'_{\alpha,1})}$$

is valid.

CHAPTER 4

POSITIVITY OF A^X AND STRUCTURE OF FRACTIONAL SPACES $E_{\alpha,P}(L_P[0,1], A^X)$

In Chapter 4, the positivity of the differential operator A^x in $L_p\,[0,1]$ is established. The structure of fractional spaces $E_{\alpha,p}(L_p[0,1], A^x)$ will be investigated. It is established that for any $0 < \alpha < 1/2$ the norms in the spaces $E_{\alpha,p}(L_p[0,1], A^x)$ and $W_p^{2\alpha}\,[0,1]$ are equivalent. This result permits us to prove the positivity of A^x in $W_p^{2\alpha}\,[0,1]\,(0 < \alpha < 1/2)$. In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for elliptic and parabolic equations.

4.1 POSITIVITY OF A^X IN $L_P\,[0,1]$

First, the positivity of A^x in $L_p\,[0,1]$ is investigated.

Theorem 4.1.1. *For all $\lambda \in R_\varphi = \{\lambda : |\arg\lambda| \le \varphi, \varphi < \pi/2\}$, the resolvent $(\lambda I + A^x)^{-1}$ defined by formula (2.4) satisfies the following estimate*

$$\left\|(\lambda I + A^x)^{-1}\right\|_{L_p[0,1]\to L_p[0,1]} \le \frac{M(\varphi,\delta)}{1+|\lambda|}.$$

Proof. Using formulas (3.1) and (3.2), we get

$$|u(x)| \le \int_0^1 |G(x,s,\mu;\lambda+\delta)|\,|f_*(s)|\,ds$$

for any $x \in [0,1]$ .Then, using estimates (2.7), (2.8) and the triangle inequality, we obtain

$$\left(\int_0^1 |u(x)|^p\,dx\right)^{\frac{1}{p}} \le \left(\int_0^1 \left(\int_0^1 |G(x,s,\mu;\lambda+\delta)|\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$= \left(\int_{\frac{1+\mu}{2}}^1 \left(\int_0^{\mu} |G(x,s,\mu;\lambda+\delta)|\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$+ \left(\int_{\frac{1+\mu}{2}}^1 \left(\int_{\mu}^1 |G(x,s,\mu;\lambda+\delta)|\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$+ \left(\int_0^{\frac{1+\mu}{2}} \left(\int_0^1 |G(x,s,\mu;\lambda+\delta)|\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$\le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left(\int_{\frac{1+\mu}{2}}^1 \left(\int_0^{\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left(\int_{\frac{1+\mu}{2}}^1 \left(\int_{\mu}^{x+\frac{\mu-1}{2}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)}\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left(\int_{\frac{1+\mu}{2}}^1 \left(\int_{x+\frac{\mu-1}{2}}^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left(\int_0^{\frac{1+\mu}{2}} \left(\int_0^1 e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\,|f_*(s)|\,ds\right)^p dx\right)^{\frac{1}{p}}$$

$$\le \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left(\int_{\frac{1+\mu}{2}}^1 \left(\int_0^{1+\mu} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-y+\mu)}\,|f_*(y-x)|\,dy\right)^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} \left(\int_{\frac{1+\mu}{2}}^1 \left(\int_{\mu-1}^{\frac{\mu-1}{2}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1+y-\mu)}\,|f_*(y+x)|\,dy\right)^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\left(\int\limits_{\frac{1+\mu}{2}}^{1}\left(\int\limits_{\frac{\mu-1}{2}}^{1}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}\left|f_*(y+x)\right|dy\right)^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\left(\int\limits_{0}^{\frac{1+\mu}{2}}\left(\int\limits_{-1}^{1}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}\left|f_*(y+x)\right|dy\right)^p dx\right)^{\frac{1}{p}}$$

$$\leq\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{0}^{1+\mu}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-y+\mu)}dy\int\limits_{\frac{1+\mu}{2}}^{1}\left|f_*(y-x)\right|dx$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\mu-1}^{\frac{\mu-1}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1+y-\mu)}dy\left(\int\limits_{\frac{1+\mu}{2}}^{1}\left|f_*(y+x)\right|^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\frac{1+\mu}{2}}^{1}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\left(\int\limits_{\frac{\mu-1}{2}}^{1}\left|f_*(y+x)\right|^p dx\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{0}^{\frac{1+\mu}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\left(\int\limits_{-1}^{1}\left|f_*(y+x)\right|^p dx\right)^{\frac{1}{p}}$$

$$\leq\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{0}^{1+\mu}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-y+\mu)}dy\left(\int\limits_{0}^{1}\left|f_*(s)\right|^p ds\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\mu-1}^{\frac{\mu-1}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1+y-\mu)}dy\left(\int\limits_{0}^{1}\left|f_*(s)\right|^p ds\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{\frac{1+\mu}{2}}^{1}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\left(\int\limits_{0}^{1}\left|f_*(s)\right|^p ds\right)^{\frac{1}{p}}$$

$$+\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}\int\limits_{0}^{\frac{1+\mu}{2}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|y|}dy\left(\int\limits_{0}^{1}\left|f_*(s)\right|^p ds\right)^{\frac{1}{p}}$$

$$\leq\frac{M(\varphi,\delta)}{1+|\lambda|}\left\|f\right\|_{L_p[0,1]}.$$

Thus, we have

$$\left\|(A^x+\lambda)^{-1}f\right\|_{L_p[0,1]}\leq\frac{M(\varphi,\delta)}{1+|\lambda|}\left\|f\right\|_{L_p[0,1]}.$$

From that it follows

$$\left\|(A^x+\lambda)^{-1}\right\|_{L_p[0,1]\to L_p[0,1]} \le \frac{M(\varphi,\delta)}{1+|\lambda|}.$$

Theorem 4.1.1 is proved.

Clearly, the operators A^x and its resolvent $(\lambda + A^x)^{-1}$ commute. Thus, from the definition of the norm in the space $E_{\alpha,p}(L_p\,[0,1]\,,A^x)$ it follows that

$$\left\|(\lambda+A^x)^{-1}\right\|_{E_{\alpha,p}(L_p[0,1],A^x)\to E_{\alpha,p}(L_p[0,1],A^x)} \le \left\|(\lambda+A^x)^{-1}\right\|_{L_p[0,1]\to L_p[0,1]}.$$

Hence, by using Theorem 4.1.1, we obtain the positivity of the operator A^x in the fractional spaces $E_{\alpha,p}(L_p\,[0,1]\,,A^x)$.

4.2 THE STRUCTURE OF FRACTIONAL SPACES $E_{\alpha,P}(L_P\,[0,1]\,,A^X)$, POSITIVITY OF A^X IN $\overset{\circ}{W}_P^{2\alpha}[0,1]$

Now, we will study the positivity of A^x in $\overset{\circ}{W}_p^{2\alpha}[0,1]$. We have the following theorem.

Theorem 4.2.1. *Let* $\alpha \in (0,1/2), 1 \le p < \infty$. *Then, the norms of the spaces* $E_{\alpha,p}(L_p\,[0,1]\,,A^x)$ *and* $\overset{\circ}{W}_p^{2\alpha}[0,1]$ *are equivalent.*

Here, $\overset{\circ}{W}_p^{2\alpha}[0,1]$ $(0<2\alpha<1)$ is the Banach space of all integrable functions $\varphi(x)$ defined on $[0,1]$ and satisfying a Hölder condition for which the following norm is finite:

$$\|\varphi\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]} = \left(\int_0^1\int_0^1 \frac{|\varphi(x+y)-\varphi(x)|^p}{y^{1+2\alpha p}}dydx\right)^{\frac{1}{p}} + \int_0^1 \frac{|\varphi(x)-\varphi(0)|}{x^{2(-1+p(\alpha+1))}}dx + \|\varphi\|_{L_p[0,1]}.$$

Proof. For any $\lambda > 0$, using formula (2.38), the triangle inequality, we obtain

$$\left(\int_0^\infty\left(\int_0^1 \lambda^\alpha\left|A^x(\lambda+A^x)^{-1}f(x)\right|dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$\le \left(\int_0^\infty\left(\int_0^1 \frac{\delta\lambda^\alpha}{\delta+\lambda}\left|f(x)\right|dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$+\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1\left[e^{-\sqrt{\delta+\lambda}x}+e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})\right.\right.\right.$$

$$\left.\left.\left.\times(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\right]|f(x)|\,dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$+\left(\int_0^\infty\left(\int_0^1\lambda^{\alpha+1}\int_0^1|G(x,s,\mu;\lambda+\delta)|\,|f(x)-f(s|\,dsdx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}},$$

$$=Q_1+Q_2+Q_3,$$

where

$$Q_1=\left(\int_0^\infty\left(\frac{\delta\lambda^\alpha}{\delta+\lambda}\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}\int_0^1|f(x)|\,dx,$$

$$Q_2=\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1\left[e^{-\sqrt{\delta+\lambda}x}+e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})\right.\right.\right.$$

$$\left.\left.\left.\times(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\right]|f(x)|\,dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}},$$

$$Q_3=\left(\int_0^\infty\left(\int_0^1\lambda^{\alpha+1}\int_0^1|G(x,s,\mu;\lambda+\delta)|\,|f(x)-f(s|\,dsdx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}.$$

Using the definition of norm space $\overset{\circ}{W}_p^{2\alpha}[0,1]$, we get

$$Q_1\le\left(\int_0^\infty\left(\frac{\delta\lambda^\alpha}{\delta+\lambda}\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}\|f\|_{L_1[0,1]}$$

$$\le M\left[\left(\int_0^1\frac{d\lambda}{\lambda^{1-\alpha p}}\right)^{\frac{1}{p}}+\delta\left(\int_1^\infty\frac{d\lambda}{\lambda^{1+(1-\alpha)p}}\right)^{\frac{1}{p}}\right]\|f\|_{L_p[0,1]}$$

$$\le M(\delta,p,\alpha)\,\|f\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]}.$$

Now, we will estimate Q_2. We have that

$$Q_2\le\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1e^{-\sqrt{\delta+\lambda}x}\,|f(x)-f(0)|\,dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$+\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1\Big[e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})\right.\right.$$

$$\left.\left.\times(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big]\,|f(x)|\,dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}},$$

$$=Q_{21}+Q_{22},$$

where

$$Q_{21}=\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1 e^{-\sqrt{\delta+\lambda}x}\,|f(x)-f(0)|\,dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}},$$

$$Q_{22}=\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}\int_0^1\Big[e^{-\sqrt{\delta+\lambda}\mu}(e^{-\sqrt{\delta+\lambda}(1-x)}-e^{-\sqrt{\delta+\lambda}(1+x)})\right.\right.$$

$$\left.\left.\times(1+e^{-\sqrt{\delta+\lambda}(1+\mu)})^{-1}\Big]\,|f(x)|\,dx\right)^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}.$$

First, we will estimate Q_{21}. We have that

$$Q_{21}\leq\int_0^1|f(x)-f(0)|\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x}\right)^p\frac{d\lambda}{\delta+\lambda}\right)^{\frac{1}{p}}dx.$$

Since

$$\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}x}\right)^p\frac{d\lambda}{\delta+\lambda}\right)^{\frac{1}{p}}\leq\left(\int_0^\infty(\delta+\lambda)^{-1+(1+\alpha)p}\,e^{-p\sqrt{\delta+\lambda}x}d\lambda\right)^{\frac{1}{p}}$$

$$=2\int_{\sqrt{\delta}x}^\infty y^{2(-1+p(\alpha+1))}e^{-y}dyx^{-2(-1+p(\alpha+1))}$$

$$\leq 2\int_0^\infty y^{2(-1+p(\alpha+1))}e^{-py}dyx^{-2(-1+p(\alpha+1))}=M(p,\alpha)x^{-2(-1+p(\alpha+1))},$$

we have that

$$Q_{21}\leq M(p,\alpha)\int_0^1 x^{-2(-1+p(\alpha+1))}\,|f(x)-f(0)|\,dx\leq M(p,\alpha)\,\|f\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]}.$$

Second, we estimate Q_{22}. We have that

$$Q_{22}\leq M(\delta)\int_0^1|f(x)|\left(\int_0^\infty\left(\frac{\lambda^{\alpha+1}}{\delta+\lambda}e^{-\sqrt{\delta+\lambda}(1+\mu-x)}\right)^p\frac{d\lambda}{\delta+\lambda}\right)^{\frac{1}{p}}dx.$$

In the same manner, we obtain

$$\left(\int_0^\infty \left(\frac{\lambda^{\alpha+1}}{\delta+\lambda} e^{-\sqrt{\delta+\lambda}(1+\mu-x)}\right)^p \frac{d\lambda}{\delta+\lambda}\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^\infty (\delta+\lambda)^{-1+(1+\alpha)p} e^{-p\sqrt{\delta+\lambda}(1+\mu-x)} d\lambda\right)^{\frac{1}{p}}$$

$$= 2 \int_{\sqrt{\delta}(1+\mu-x)}^{\infty} y^{2(-1+p(\alpha+1))} e^{-y} dy (1+\mu-x)^{-2(-1+p(\alpha+1))}$$

$$\leq 2 \int_0^\infty y^{2(-1+p(\alpha+1))} e^{-py} dy (1+\mu-x)^{-2(-1+p(\alpha+1))}$$

$$= M_1(p,\alpha)(1+\mu-x)^{-2(-1+p(\alpha+1))}.$$

Therefore,

$$Q_{22} \leq M_1(p,\alpha) \int_0^1 (1+\mu-x)^{-2(-1+p(\alpha+1))} |f(x)|\, dx \leq M_2(p,\alpha) \|f\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]}.$$

Then,

$$Q_2 \leq M_3(p,\alpha) \|f\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]}.$$

Now, we will estimate Q_3.

$$Q_3 = \left(\int_0^\infty \left(\int_0^1 \lambda^{\alpha+1} \int_0^1 |G(x,s,\mu;\lambda+\delta)|\, |f(x)-f(s|\, ds dx\right)^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$\leq \left(\int_0^{\frac{1+\mu}{2}} \int_0^1 |f(x)-f(s)|\, ds dx \int_0^\infty \left(\lambda^{\alpha+1} \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}|x-s|}\right)^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$+ \left(\int_{\frac{1+\mu}{2}}^1 \int_0^\mu |f(x)-f(s)|\, ds dx \int_0^\infty \left(\lambda^{\alpha+1} \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(1-x+\mu-s)}\right)^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$+ \int_{\frac{1+\mu}{2}}^1 \int_\mu^{x+\frac{\mu-1}{2}} |f(x)-f(s)|\, ds dx \left(\int_0^\infty \left(\lambda^{\alpha+1} \frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}} e^{-\left(\delta^2+|\lambda|^2\right)^{1/4} \frac{\sqrt{2}}{2}(1-x+s-\mu)}\right)^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

$$+\int\limits_{\frac{1+\mu}{2}}^{1}\int\limits_{x+\frac{\mu-1}{2}}^{1}|f(x)-f(s)|\,dsdx\left(\int\limits_{0}^{\infty}\left(\lambda^{\alpha+1}\frac{M_1(\delta,\mu)}{\left(\delta^2+|\lambda|^2\right)^{1/4}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\right)^{\frac{1}{p}}\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}.$$

We have that

$$\int\limits_{0}^{\infty}\left(\lambda^{\alpha+1}\frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/4}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}|x-s|}\right)^{p}\frac{d\lambda}{\lambda}$$

$$\leq\int\limits_{0}^{\infty}\frac{1}{\lambda^{1-p\left(\frac{1}{2}+\alpha\right)}}e^{-\lambda^{\frac{1}{2}}\frac{\sqrt{2}}{2}|x-s|}d\lambda.$$

Putting $\lambda^{\frac{1}{2}}\frac{\sqrt{2}}{2}|x-s|=y$, we get $\lambda=\frac{2y^2}{|x-s|^2}$ and $d\lambda=\frac{4ydy}{|x-s|^2}$. Then

$$\int\limits_{0}^{\infty}\frac{1}{\lambda^{1-p\left(\frac{1}{2}+\alpha\right)}}e^{-p\lambda^{\frac{1}{2}}\frac{\sqrt{2}}{2}|x-s|}d\lambda=\int\limits_{0}^{\infty}\left(\frac{2y^2}{|x-s|^2}\right)^{1-p\left(\frac{1}{2}+\alpha\right)}e^{-py}\frac{4ydy}{|x-s|^2}$$

$$=4.2^{p\left(\frac{1}{2}+\alpha\right)}\frac{1}{|x-s|^{-2+2p\left(\frac{1}{2}+\alpha\right)+2}}\int\limits_{0}^{\infty}e^{-py}y^{p\left(\frac{1}{2}+\alpha\right)-1}dy=M(\alpha,p)\frac{1}{|x-s|^{p(2\alpha+1)}}$$

and

$$\int\limits_{0}^{\infty}\frac{1}{\lambda^{1-p\left(\frac{1}{2}+\alpha\right)}}e^{-p\lambda^{\frac{1}{2}}\frac{\sqrt{2}}{2}|x-s|}d\lambda=M(\alpha,p)\frac{1}{|x-s|^{p(2\alpha+1)}}.$$

Likewise, we can show that

$$\int\limits_{0}^{\infty}\left(\lambda^{\alpha+1}\frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/4}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+s-\mu)}\right)^{p}\frac{d\lambda}{\lambda}$$

$$\leq M(\alpha,p)\frac{1}{(1-x+s-\mu)^{p(2\alpha+1)}}$$

and

$$\int\limits_{0}^{\infty}\left(\lambda^{\alpha+1}\frac{1}{\left(\delta^2+|\lambda|^2\right)^{1/4}}e^{-\left(\delta^2+|\lambda|^2\right)^{1/4}\frac{\sqrt{2}}{2}(1-x+\mu-s)}\right)^{p}\frac{d\lambda}{\lambda}$$

$$\leq M(\alpha,p)\frac{1}{(1-x+\mu-s)^{p(2\alpha+1)}}.$$

Thus,

$$Q_3\leq M(\alpha,p)\int\limits_{0}^{\frac{1+\mu}{2}}\int\limits_{0}^{1}\frac{|f(x)-f(s)|}{|x-s|^{2\alpha+1}}dsdx$$

$$+M(\alpha,p)\int\limits_{\frac{1+\mu}{2}}^{1}\int\limits_{0}^{\mu}\frac{|f(x)-f(s)|}{(1-x+\mu-s)^{2\alpha+1}}dsdx$$

$$+M(\alpha,p)\int\limits_{\frac{1+\mu}{2}}^{1}\int\limits_{\mu}^{x+\frac{\mu-1}{2}}\frac{|f(x)-f(s)|}{(1-x+s-\mu)^{2\alpha+1}}dsdx$$

$$+M(\alpha,p)\int\limits_{\frac{1+\mu}{2}}^{1}\int\limits_{x+\frac{\mu-1}{2}}^{1}\frac{|f(x)-f(s|}{|x-s|^{2\alpha+1}}dsdx$$

$$=M(\alpha,p)\left\{\int\limits_{0}^{\frac{1+\mu}{2}}\int\limits_{0}^{1}\frac{|f(x)-f(s)|}{|x-s|^{2\alpha+1}}dsdx\right.$$

$$+\int\limits_{\frac{1+\mu}{2}}^{1}\int\limits_{0}^{\mu}\frac{|f(x)-f(s)|}{(1-x+\mu-s)^{2\alpha+1}}dsdx+\int\limits_{\frac{1+\mu}{2}}^{1}\int\limits_{\mu}^{x+\frac{\mu-1}{2}}\frac{|f(x)-f(s)|}{(1-x+s-\mu)^{2\alpha+1}}dsdx$$

$$\left.+\int\limits_{\frac{1+\mu}{2}}^{1}\int\limits_{x+\frac{\mu-1}{2}}^{1}\frac{|f(x)-f(s|}{|x-s|^{2\alpha+1}}dsdx\right\}$$

$$\leq M_3(\alpha,p)\left\|f\right\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]}.$$

Hence,

$$\|f\|_{E_{\alpha,p}(L_p[0,1],A^x)}\leq M_4(a,\mu,\delta)\left\|f\right\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]}.$$

Now, let us prove the reverse inequality

$$\|f\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]}\leq M_4(a,\mu,\delta)\left\|f\right\|_{E_{\alpha,p}(L_p[0,1],A^x)}. \tag{4.1}$$

Applying formula (2.44), we get

$$f(x+\tau)-f(x)$$

$$=\int\limits_0^\infty\lambda^{-\alpha}\int\limits_0^1\left[G(x+\tau,s,\mu;\lambda+\delta)-G(x,s,\mu;\lambda+\delta)\right]\lambda^\alpha A^s(\lambda+A^s)^{-1}f(s)dsd\lambda.$$

The proof estimate (4.1) is based on this formula and estimates (2.7), (2.8), and the triangle inequality. Theorem 4.2.1 is proved.

From positivity of A^x in $E_{\alpha,p}$ and Theorem 4.2.1 it follows the positivity of A^x on $\overset{\circ}{W}_p^{2\alpha}[0,1]$.

Theorem 4.2.2. *The operator* $(\lambda+A^x)$ *has a bounded inverse in* $\overset{\circ}{W}_p^{2\alpha}[0,1]$ *for any* $\lambda\geq 0$ *and the following estimate holds:*

$$\left\|(\lambda+A^x)^{-1}\right\|_{\overset{\circ}{W}_p^{2\alpha}[0,1]\to\overset{\circ}{W}_p^{2\alpha}[0,1]}\leq\frac{M(\delta)}{2\alpha(1-2\alpha)}\frac{M}{\delta+\lambda}.$$

4.3 APPLICATIONS

In applications, we will obtain new coercive inequalities for the solution of local and nonlocal boundary value problems for parabolic and elliptic equations.

4.3.1 Parabolic problems

First, we consider the initial boundary value problem (2.49).

Theorem 4.3.1. *Let* $0 < 2\alpha < 1$. *Then, for the solution of the initial value problem (2.49), we have the following coercive stability inequality*

$$\|u_t\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2\alpha}[0,1]\right)} + \|u\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2+2\alpha}[0,1]\right)}$$

$$\leq M(\alpha)\left[\|\varphi\|_{\overset{\circ}{W}_p^{2+2\alpha}[0,1]} + \|f\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2\alpha}[0,1]\right)}\right].$$

The proof of Theorem 4.3.1 is based on Theorem 4.2.1 on the structure of the fractional spaces $E_{\alpha,p} = E_{\alpha,p}\left(L_p\left[0,1\right], A^x\right)$, Theorem 4.1.1 on the positivity of the operator A^x on the following theorem on coercive stability of initial value for the abstract parabolic equation.

Theorem 4.3.2. *Let* A *be a strongly positive operator in a Banach space* E *and* $\varphi \in D\left(A\right),\ f \in L_p\left([0,T], E_{\alpha,p}\right), 0 < \alpha < 1$. *Then, for the solution of the initial value problem (2.50), the following coercive inequality holds:*

$$\|u'\|_{L_p([0,T],E_{\alpha,p})} + \|Au\|_{L_p([0,T],E_{\alpha,p})}$$

$$\leq M\left[\|A\varphi\|_{E_{\alpha,p}} + \frac{M}{\alpha(1-\alpha)}\|f\|_{L_p([0,T],E_{\alpha,p})}\right].$$

Second, we consider the nonlocal boundary value problem for parabolic equation (2.51).

Theorem 4.3.3. *Let* $0 < 2\alpha < 1$. *Then, for the solution of boundary value problem (2.51), the following coercive stability inequality*

$$\|u_t\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2\alpha}[0,1]\right)} + \|u\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2+2\alpha}[0,1]\right)} \leq M(\alpha)\|f\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2\alpha}[0,1]\right)}$$

is valid.

The proof of Theorem 4.3.3 is based on Theorem 4.2.1 on the structure of the fractional spaces $E_{\alpha,p} = E_{\alpha,p}\left(L_p\left[0,1\right], A^x\right)$,

Theorem 4.1.1 on the positivity of the operator A^x on the following theorem on the coercive stability of the nonlocal boundary value for the abstract parabolic equation.

Theorem 4.3.4. *Let A be a strongly positive operator in a Banach space E and $f \in L_p\left(\left[0,T\right], E_{\alpha,p}\right), 0 < \alpha < 1$. Then, the solution of the nonlocal boundary value problem(2.52) satisfies the following coercive inequality*

$$\|u'\|_{L_p([0,T],E_{\alpha,p})} + \|Au\|_{L_p([0,T],E_{\alpha,p})} \leq \frac{M}{\alpha(1-\alpha)} \|f\|_{L_p([0,T],E_{\alpha,p})}.$$

4.3.2 Elliptic problems

First, we consider the boundary value problem (2.53).

Theorem 4.3.5. *Let $0 < 2\alpha < 1$. Then, for the solution of the boundary value problem (2.53), the following coercive stability inequality holds:*

$$\|u_{tt}\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2\alpha}[0,1]\right)} + \|u\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2+2\alpha}[0,1]\right)}$$

$$\leq M(\alpha)\left[\|\varphi\|_{\overset{\circ}{W}_p^{2+2\alpha}[0,1]} + \|\psi\|_{\overset{\circ}{W}_p^{2+2\alpha}[0,1]} + \|f\|_{L_p\left([0,T],\overset{\circ}{W}_p^{2\alpha}[0,1]\right)}\right].$$

The proof of Theorem 4.3.5 is based on Theorem 4.2.1 on the structure of fractional spaces $E_{\alpha,p} = E_{\alpha,p}\left(L_p\left[0,1\right], A^x\right)$, Theorem 4.1.1 on the positivity of the operator A^x, on the following theorems on coercive stability of boundary value problem for the abstract elliptic equation and on the structure of the fractional space $E'_{\alpha,p} = E_{\alpha,p}(E, A^{1/2})$ which is the Banach space consisting of those $v \in E$ for which the norm

$$\|v\|_{E'_{\alpha,p}} = \left(\int_0^\infty \left(\lambda^\alpha \left\|A^{1/2}\left(\lambda + A^{1/2}\right)^{-1} v\right\|_E\right)^p \frac{d\lambda}{\lambda}\right)^{\frac{1}{p}}$$

is finite.

Theorem 4.3.6. *(Ashyralyev and Sobolevskii, 2004) The spaces $E_{\alpha,p}(E, A)$ and $E'_{2\alpha,p}(E, A^{1/2})$ coincide for any $0 < \alpha < \frac{1}{2}$, and their norms are equivalent.*

Theorem 4.3.7. *(Ashyralyev and Sobolevskii, 2004) Let A be positive operator in a Banach space E and* $\varphi \in D(A), \psi \in D(A),\ f \in L_p([0,T], E'_{\alpha,p}), 0 < \alpha < 1$. *Then, for the solution of the boundary value problem (2.54), we have the following coercive inequality*

$$\|u''\|_{L_p([0,T],E'_{\alpha,p})} + \|Au\|_{L_p([0,T],E'_{\alpha,p})}$$
$$\leq M\left[\|A\varphi\|_{E'_{\alpha,p}} + \|A\psi\|_{E'_{\alpha,p}} + \frac{M}{\alpha(1-\alpha)}\|f\|_{L_p([0,T],E'_{\alpha,p})}\right].$$

Second, we consider the nonlocal boundary value problem for the elliptic equation (2.55).

Theorem 4.3.8. *Let* $0 < 2\alpha < 1$. *Then, the solution of nonlocal boundary value problem (2.55) satisfies the following coercive stability inequality*

$$\|u_{tt}\|_{L_p\left([0,T],\overset{\circ}{W}{}_p^{2\alpha}[0,1]\right)} + \|u\|_{L_p\left([0,T],\overset{\circ}{W}{}_p^{2+2\alpha}[0,1]\right)} \leq M(\alpha)\|f\|_{L_p\left([0,T],\overset{\circ}{W}{}_p^{2\alpha}[0,1]\right)}.$$

The proof of Theorem 4.3.8 is based on Theorem 4.2.1 on the structure of the fractional spaces $E_{\alpha,p} = E_{\alpha,p}(L_p[0,1], A^x)$, Theorem 4.1.1 on the positivity of the operator A^x, Theorem 4.3.6 on the structure of the fractional space $E'_{\alpha,p} = E_{\alpha,p}(E, A^{1/2})$ and on the following theorem on coercive stability of the nonlocal boundary value problem for the abstract elliptic equation.

Theorem 4.3.9. *(Ashyralyev, 2003) Let A be a positive operator in a Banach space E and* $f \in L_p([0,T], E'_{\alpha,p})$, $0 < \alpha < 1$. *Then, for the solution of the nonlocal boundary value problem (2.56), the coercive inequality*

$$\|u''\|_{L_p([0,T],E'_{\alpha,p})} + \|Au\|_{L_p([0,T],E'_{\alpha,p})} \leq \frac{M}{\alpha(1-\alpha)}\|f\|_{L_p([0,T],E'_{\alpha,p})}$$

is valid.

CHAPTER 5

CONCLUSION

This thesis is devoted to study of second order the positive differential operator. The following original results are obtained:

- Green's function of the second order differential operator with the nonlocal condition is constructed.
- The positivity of the second order differential operator with the nonlocal condition is established.
- The structure of fractional spaces generated by this differential operator is investigated.
- In applications, theorems on well-posedness of local and nonlocal boundary value problems for parabolic and elliptic equations are established.

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APPENDIX A

DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS, FURTHER STUDIES AND PUBLICATIONS FROM THESIS WORK

A.1 DECLARATION STATEMENT FOR THE ORIGINALITY OF THE THESIS

I hereby declare that this thesis comprises my original work. No material in this thesis has been previously published and written by another person, except where due reference is made in the text of the thesis. I further declare that this thesis contains no material which has been submitted for a degree or diploma or other qualifications at any other university.

Signature:
Date: June 27, 2016

A.2 PUBLICATIONS FROM THESIS WORK

1. Allaberen Ashyralyev, and Nese Nalbant, “Positivity of a Differential Operator with Nonlocal Conditions”, AIP Conference Proceedings, (editors) Ashyralyev A, Malkowsky E , Vol. 1676, No. 020066, Oct. 2015, pp. 5. doi: 10.1063/1.4930492.

2. Allaberen Ashyralyev, and Nese Nalbant, “ Positivity of a Differential Operator with Nonlocal Conditions”, Filomat,Volume 30, No. 3, 2016, doi: 10.2298/FIL1603885A.

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M.S., Mathematics, Fatih University, Istanbul, Turkey, June 2011
Thesis: “Structure of the Fractional Spaces Generated by Positive Operators and Applications”

B.S., Mathematics, Marmara University, Istanbul, Turkey, June 1995

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- Allaberen Ashyralyev,Yasar Sozen and Nese Nalbant,The positivity of differential operator with nonlocal boundary conditions, AIP Conf. Proc. 1479, 578 (2012); doi: 10.1063/1.4756198

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